MCS 119 Exam 2 Solutions

1. (10 pts) Find the n-th derivative of
$$f(x) = \frac{1}{x}$$
by calculating the first few derivatives and observing the pattern that occurs.

$$\begin{aligned}
\frac{d}{dx}\frac{1}{x} &= \frac{d}{dx}x^{-1} \\
&= (-1)x^{-1-1} \\
&= (-1)x^{-2} \\
\frac{d^2}{dx^2}x^{-1} &= \frac{d}{dx}[(-1)x^{-2}] \\
&= (-1)\frac{d}{dx}x^{-2} \\
&= (-1)(-2)x^{-2-1} \\
&= (-1)(-2)x^{-3} \\
\frac{d^3}{dx^3}x^{-1} &= \frac{d}{dx}[(-1)(-2)x^{-3}] \\
&= (-1)(-2)\frac{d}{dx}x^{-3} \\
&= (-1)(-2)(-3)x^{-3-1} \\
&= (-1)(-2)(-3)x^{-4} \\
\frac{d^4}{dx^4}x^{-1} &= \frac{d}{dx}[(-1)(-2)(-3)x^{-4}] \\
&= (-1)(-2)(-3)\frac{d}{dx}x^{-4} \\
&= (-1)(-2)(-3)(-4)x^{-4-1} \\
&= (-1)(-2)(-3)(-4)x^{-5}
\end{aligned}$$

Now, the pattern is quite clear. The exponent on the x decreases by 1 every time we differentiate. So after differentiating n times, it will be $-1-n$ or $-n-1$. The coefficient also shows a simple pattern: it is $(-1)(-2)(-3)\cdots(-n)$. So
$$\frac{d^n}{dx^n}x^{-1} = (-1)(-2)(-3)\cdots(-n)x^{-n-1}.$$
If you want to express this is a more compact form, then you can write
$$(-1)(-2)(-3)\cdots(-n) = (-1)^n 1\cdot 2\cdot 3\cdots n = (-1)^n n!$$
Hence
$$\frac{d^n}{dx^n}x^{-1} = (-1)^n n! x^{-n-1}.$$

2. If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) (4 pts) $y = \frac{x^2}{f(x)}$

By the Quotient Rule,

$$y' = \frac{x^2}{f(x)} = \frac{\left(\frac{d}{dx}x^2\right) f(x) - x^2 \frac{d}{dx} f(x)}{[f(x)]^2} = \frac{2xf(x) - x^2 f'(x)}{f^2(x)}$$

(b) (6 pts) $y = \frac{1+xf(x)}{\sqrt{x}}$

The easiest way to do this is to simplify first:

$$\frac{1+xf(x)}{\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{xf(x)}{\sqrt{x}} = x^{-1/2} + x^{-1/2} f(x).$$

Now,

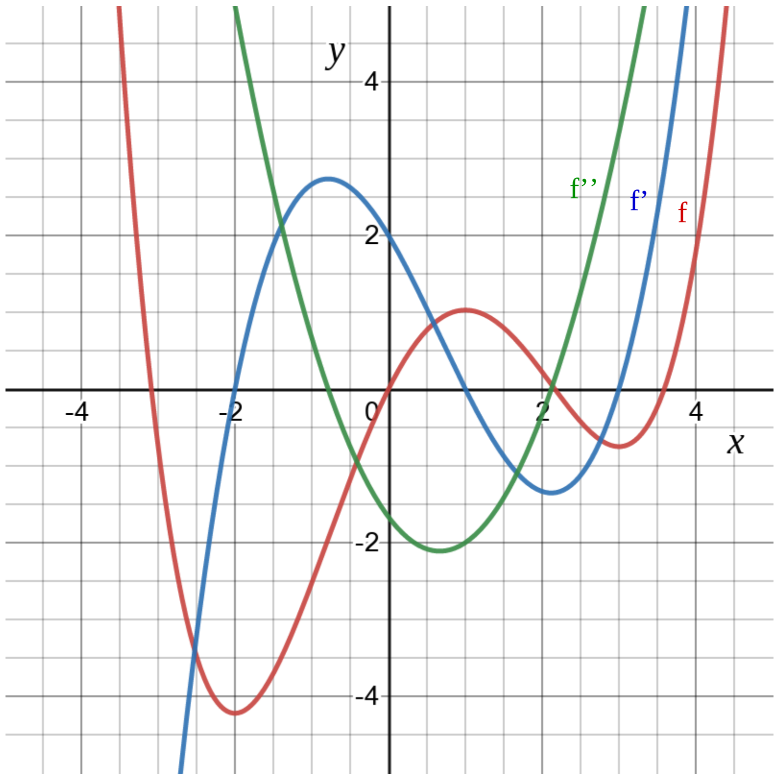
$$\begin{aligned} y' &= \frac{d}{dx} x^{-1/2} + \frac{d}{dx}\left[x^{1/2} f(x)\right] \\ &= -\frac{1}{2} x^{-1/2-1} - \frac{1}{2} x^{1/2-1} f(x) + x^{-1/2} f'(x) \\ &= -\frac{1}{2} x^{-3/2} + \frac{1}{2} x^{-1/2} f(x) + x^{1/2} f'(x) \end{aligned}$$

by the Product Rule.

Alternately, you can use the Quotient Rule and the Product Rule:

$$\begin{aligned} y' &= \frac{\frac{d}{dx}\big(1+xf(x)\big)\sqrt{x} - \big(1+xf(x)\big)\frac{d}{dx}\sqrt{x}}{\sqrt{x}^2} && \text{by the Quotient Rule} \\ &= \frac{\left(\frac{d}{dx}x\, f(x) + x\frac{df}{dx}\right)\sqrt{x} - \big(1+xf(x)\big)\frac{d}{dx}x^{1/2}}{x} && \text{by the Product Rule} \\ &= \frac{\big(f(x) + xf'(x)\big)\sqrt{x} - \big(1+xf(x)\big)\frac{1}{2}x^{-1/2}}{x} \\ &= \frac{\big(f(x) + xf'(x)\big)\sqrt{x} - \big(1+xf(x)\big)\frac{1}{2\sqrt{x}}}{x} \\ &= \frac{2x\big(f(x) + xf'(x)\big) - \big(1+xf(x)\big)}{2x\sqrt{x}} \\ &= \frac{2x(f(x) + 2x^2 f'(x) - 1 - xf(x)}{2x\sqrt{x}} \\ &= \frac{x(f(x) + 2x^2 f'(x) - 1}{2x\sqrt{x}} \\ &= \frac{1}{2}x^{-1/2} f(x) + x^{1/2} f'(x) - \frac{1}{2}x^{-3/2} \end{aligned}$$

3. The graph of the function f is given below.



(a) (7 pts) Sketch the graphs of f' and f''. You may do this on the same axes above.

This can be done using the same method we used in class. Just slide the tangent line along the graph of f and plot its slope along the way. This will give you the graph of f'. Then do the same thing to the graph of f' to plot the graph of f''.

(b) (3 pts) For what values of x is $f'(x) = 0$? What can you say about f at those points?

The places where $f'(x) = 0$ are those where the tangent line to f is horizontal. It is easy to read these off the graph of f: they are $x = -2$, $x = 1$, and $x = 3$. You can easily read these off the graph of f' too if you drew that graph well.

4. (a) (3 pts) Let f be a function of real numbers and let a be a real number. What does it mean for f to be differentiable at a?

It means that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.

(b) (7 pts) Give a rigorous argument why the absolute value function $f(x) = |x|$ is not differentiable at 0.

The absolute value function has a cusp at $x = 0$. Here is the way argued in class why its derivative does not exist.

$$f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

Looking at the left-hand limit,

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

by LL7 and because $|x| = x$ for any positive number x. Looking at the right-hand limit,

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

by LL7 and because $|x| = -x$ for any negative number x. Since the one-sided limits are not equal, the two-sided limit does not exist. Hence $f(x) = |x|$ is not diffrerentiable at $x = 0$.

5. (10 pts)**Extra credit problem.** Let f be a function of real numbers. The *symmetric derivative* of f at $x \in \mathbb{R}$ is defined as

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

If the symmetric derivative of f exists at x, then f is said to be *symmetrically differentiable* at x.

Prove that the absolute value function $f(x) = |x|$ is symmetrically differentiable at 0 by showing that its symmetric derivative exists there.

The symmetric derivative of $f(x) = |x|$ at 0 is

$$\begin{aligned}
\lim_{h\to 0} \frac{|0+h| - |0-h|}{2h} &= \lim_{h\to 0} \frac{|h| - |-h|}{2h} \\
&= \lim_{h\to 0} \frac{0}{2h} && \text{since } |h| = |-h| \\
&= \lim_{h\to 0} 0 \\
&= 0 && \text{by LL7}
\end{aligned}$$

Since its symmetric derivative exists at $x = 0$, the function $f(x) = |x|$ is symmetrically differentiable at 0.