MCS 119 Final Exam Solutions

May 11, 2021

1. (10 pts) Let
$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$
Find the horizontal aysmptotes of f by calculating the exact values of the limits
$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \quad \text{and} \quad \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}.$$
Did you get the same value or different values for these two limits?

Since $x \to \infty$, x can be treated as a large positive number, and so we do not need to be concerned about what happens when $x = 0$. Therefore we can divide the numerator and the denominator by x:
$$\frac{\sqrt{2x^2+1}}{3x-5} = \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}} = \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{\frac{2x^2+1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}},$$
where we used the fact that $x = \sqrt{x^2}$ since x is positive. We know that
$$\lim_{x\to\infty} \frac{1}{x^2} = 0,$$
and hence
$$\lim_{x\to\infty} \left(2 + \frac{1}{x^2}\right) = 2 + 0 = 2$$
by LL1 and LL7, and finally,
$$\lim_{x\to\infty} \sqrt{2+\frac{1}{x^2}} = \sqrt{\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)} = \sqrt{2}$$
by LL11. Similarly,
$$\lim_{x\to\infty} \frac{1}{x} = 0 \implies \lim_{x\to\infty}\left(3 - \frac{5}{x}\right) = \lim_{x\to\infty} 3 - 5 \lim_{x\to\infty} \frac{1}{x} = 3 - 0 = 3$$
by LL2, LL3, and LL7. Therefore by LL5,
$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim_{x\to\infty}\left(3-\frac{5}{x}\right)} = \frac{\sqrt{2}}{3}.$$

The other limit can be calculated much the same way, except that when x is negative, $\sqrt{x^2} = |x| = -x$, and so
$$\frac{\sqrt{2x^2+1}}{x} = -\frac{\sqrt{2x^2+1}}{-x} = \frac{\sqrt{2x^2+1}}{\sqrt{x^2}} = -\sqrt{2+\frac{1}{x^2}}.$$
The rest of the calculation is exactly the same. Hence the limit differs only by the sign:
$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\lim_{x\to\infty}\left(-\sqrt{2+\frac{1}{x^2}}\right)}{\lim_{x\to\infty}\left(3-\frac{5}{x}\right)} = -\frac{\sqrt{2}}{3}.$$

Therefore the horizontal asymptotes are $y = \sqrt{2}/3$ and $y = -\sqrt{2}/3$.

Here is another way the limit as $x \to \infty$ can be calculated, which is less rigorous but is still convincing enough. If x is a large number, then

$$2x^2 + 1 \approx 2x^2 \quad \text{and} \quad 3x - 5 \approx 3x.$$

The relative error (compared to the size of the number) in these estimates gets smaller and smaller as x gets larger because the $2x^2$ and the $3x$ dominate compared to the 1 and the -5. So the estimates are getting better and better as $x \to \infty$. In fact, the estimates are perfect in the limit. Therefore

$$\frac{\sqrt{2x^2+1}}{3x-5} \approx \frac{\sqrt{2x^2}}{3x}$$

when x is large. This estimate also gets better and better as $x \to \infty$ and is perfect in the limit. Hence

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2x^2}}{3x}.$$

Since $x \to \infty$, we can treat x as a (large) positive number. So

$$\sqrt{2x^2} = \sqrt{2}\sqrt{x^2} = \sqrt{2}|x| = \sqrt{2}x,$$

and

$$\frac{\sqrt{2x^2}}{3x} = \frac{\sqrt{2}x}{3x} = \frac{\sqrt{2}}{3}$$

because we can cancel x as $x \neq 0$. Finally, by LL7,

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2}}{3} = \frac{\sqrt{2}}{3}$$

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The limit as $x \to -\infty$ can be evaluated similarly, keeping in mind that $|x| = -x$ when x is negative.

2. (10 pts) Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is the line with equation

$$\frac{x_0}{a^2}\, x + \frac{y_0}{b^2}\, y = 1$$

By implicit differentiation,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) &= \frac{d}{dx}1 \\
\frac{2x}{a^2} + \frac{2y\frac{dy}{dx}}{b^2} &= 0 \\
\frac{2y\frac{dy}{dx}}{b^2} &= -\frac{2x}{a^2} \\
\frac{y}{b^2}\frac{dy}{dx} &= -\frac{x}{a^2} \\
\frac{dy}{dx} &= -\frac{b^2 x}{a^2 y}
\end{aligned}$$

So at (x_0, y_0),

$$\frac{dy}{dx} = -\frac{b^2 x_0}{a^2 y_0},$$

and the equation of the tangent line is

$$y = -\frac{b^2 x_0}{a^2 y_0}\,x + c.$$

Since the tangent line passes through the point (x_0, y_0), we can find c by substituting (x_0, y_0) into the equation of the line:

$$\begin{aligned} y_0 &= -\frac{b^2 x_0}{a^2 y_0}\,x_0 + c \\ c &= y_0 + \frac{b^2 x_0}{a^2 y_0}\,x_0 = \frac{a^2 y_0^2}{a^2 y_0} + \frac{b^2 x_0^2}{a^2 y_0} = \frac{a^2 y_0^2 + b^2 x_0^2}{a^2 y_0} \end{aligned}$$

We also know that the point (x_0, y_0) is on the ellipse, so it satisfies the equation of the ellipse:

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \implies b^2 x_0^2 + a^2 y_0^2 = a^2 b^2.$$

Substituting this into the numerator of the expression for c gives

$$c = \frac{a^2 b^2}{a^2 y_0} = \frac{b^2}{y_0}.$$

So the equation of the tangent line is

$$y = -\frac{b^2 x_0}{a^2 y_0}\,x + \frac{b^2}{y_0}.$$

To make this more symmetric, we can multiply by $\frac{y_0}{b^2}$:

$$\frac{y_0}{b^2}\,y = -\frac{x_0}{a^2}\,x + 1.$$

Finally, we can add $\frac{x_0}{a^2}\,x$ to both sides:

$$\frac{y_0}{b^2}\,y + \frac{x_0}{a^2}\,x = 1$$

which is what we wanted to show.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \frac{x^2 - 4}{x^2 + 4}.$$

(a) (8 pts) Find the intervals of concavity, that is those intervals on which f is concave up and those on which f is concave down.

The graph is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$. By the Quotient Rule,

$$f'(x) = \frac{2x(x^2+4) - 2x(x^2-4)}{(x^2+4)^2} = \frac{2x^3 + 8x - 2x^3 + 8x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}.$$

Using the Quotient Rule and the Chain Rule,

$$\begin{aligned} f''(x) &= \frac{d}{dx}\frac{16x}{(x^2+4)^2} \\ &= \frac{16(x^2+4)^2 - 16x2(x^2+4)(2x)}{(x^2+4)^4} \\ &= \frac{16(x^2+4) - 64x^2}{(x^2+4)^3} \\ &= \frac{16x^2+64-64x^2}{(x^2+4)^3} \\ &= \frac{64-48x^2}{(x^2+4)^3} \end{aligned}$$

Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, we know x^2+4 is always positive, and hence $(x^2+4)^3$ is always positive. Therefore the sign of f'' is the same as the sign of the numerator above. So $f''(x) > 0$ whenever

$$64 - 48x^2 > 0 \implies 64 > 48x^2 \implies \frac{4}{3} > x^2 \implies -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$$

Hence $f''(x) > 0$ on the interval $(-2/\sqrt{3}, 2/\sqrt{3})$, and hence f is concave up there. Similarly, $f''(x) < 0$ whenever

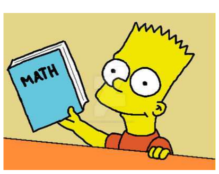
$$64 - 48x^2 < 0 \implies 64 < 48x^2 \implies \frac{4}{3} < x^2 \implies -\frac{2}{\sqrt{3}} > x \text{ or } x > \frac{2}{\sqrt{3}}.$$

So $f''(x) < 0$ and hence f is concave down on $(-\infty, -2/\sqrt{3})$ and on $(2/\sqrt{3}, \infty)$.

(b) (2 pts) What are the inflection points of f? How do you know these are inflection points?

As we saw in part (a), f'' changes sign from negative to positive or vice versa, at $x = -2/\sqrt{3}$ and at $x = 2/\sqrt{3}$, and so these are the inflection points of f.

4. (a) (4 pts) When asked to use the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$ at $x = 1/2$, Bart Simpson, who is not much of a calculus whiz, wrote the following:



$$f'\left(\frac{1}{2}\right) = \lim_{h\to 0}\frac{\sqrt{1/2+h}-\sqrt{1/2}}{h} = \frac{\sqrt{1/2}+\sqrt{h}-\sqrt{1/2}}{h} = \frac{\sqrt{h}}{h} = \frac{0}{0} = 1$$

Find all of Bart's mistakes and explain why they are wrong.

Bart made four mistakes. First, he dropped the limit after the second $=$ sign. Second, he wrote $\sqrt{1/2+h} = \sqrt{1/2}+\sqrt{h}$, which is not true as $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$ in general. Third, he tried to use LL5 to evaluate

$$\lim_{h\to 0}\frac{\sqrt{h}}{h} = \frac{\lim_{h\to 0}\sqrt{h}}{\lim_{h\to 0} h}$$

But LL5 does not work here because the limit in the denominator is 0. Fourth, Bart claimed $0/0 = 1$, which is false because $0/0$ is not defined. And of course, in general, Bart did not justify his work.

(b) (6 pts) Use the definition of the derivative (correctly) to find the derivative of $f(x) = \sqrt{x}$ for any real number $x > 0$.

Let $a > 0$. By the definition of the derivative,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}.$$

Since a is positive, x is also positive when it is sufficiently close to a. Therefore $\sqrt{a} > 0$ and $\sqrt{x} > 0$ and hence $\sqrt{x} + \sqrt{a} > 0$. Since $\sqrt{x} + \sqrt{a} \neq 0$, we can multiply both the numerator and the denominator by it

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{\sqrt{x}^2 - \sqrt{a}^2}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})}.$$

As $x \to a$, x is close to a but not equal to a, so $x - a \neq 0$. Hence we can cancel $x - a$.

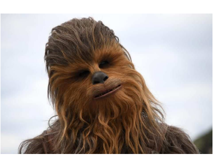
$$\frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}.$$

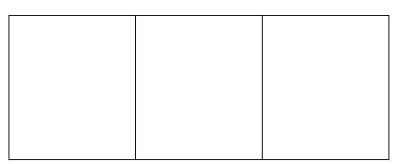
Now

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} \\ &= \frac{\lim_{x \to a} 1}{\lim_{x \to a}(\sqrt{x} + \sqrt{a})} && \text{by LL5} \\ &= \frac{1}{\lim_{x \to a} \sqrt{x} + \lim_{x \to a} \sqrt{a}} && \text{by LL7 and LL1} \\ &= \frac{1}{\sqrt{a} + \sqrt{a}} && \text{by LL10 and LL7} \\ &= \frac{1}{2\sqrt{a}} \end{aligned}$$

Hence $f'(x) = \frac{1}{2\sqrt{x}}$.

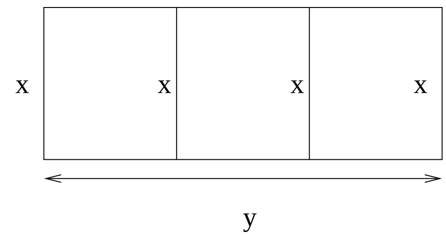
5. (10 pts) In recognition of his highly advanced engineering skills, Chewbacca is appointed to design a holding pen for the rebels' tauntauns on the planet Hoth. The holding pen is to be rectangular in shape, divided into three equal size rectangles by two rows of fencing parallel to one of the side, as in the diagram:





The reason the pen is to be divided into three parts has something to do with tauntaun biology. Chewbacca is not an expert at tauntaun biology, so instead of getting hung up on this detail, he would rather focus on designing the pen. He has 420 meters of fencing material available to him, and wants to build the holding pen with the largest total area possible. What is the largest possible area of such a holding pen? How do you know this is the absolute maximum?

Let x be the length of the four parallel sides and y be the length of the two parallel sides:



The total area of the holding pen is $A = xy$. It is clear from the diagram that the length of fencing material needed to build it is $4x + 2y$. We are limited by $2x + 4y = 420$. We can use this to express y in terms of x:

$$4x + 2y = 420 \implies 2y = 420 - 4x \implies y = 210 - 2x.$$

So

$$A = x(210 - 2x) = 210x - 2x^2.$$

Since A is a polynomial function of x, it is continuous. We know $x \geq 0$. But y cannot be negative either, and so

$$0 \leq y = 210 - 2x \implies 2x \leq 210 \implies x \leq 105.$$

So we are trying to find the absolute maximum of the continuous function $A(x) = 210x - 2x^2$ over the closed interval $x \in [0, 210]$. We an use the closed-interval method. The Extreme Value Theorem guarantees that A has an absolute maximum on the closed interval $[0, 210]$. If the absolute maximum is an interior point of the interval then it is also a local maximum and hence by Fermat's Theorem, a critical point. Otherwise the absolute maximum could be at one of the endpoints. So we will find the critical points of A. Since A is a polynomial, it is differentiable everywhere. Therefore the only kind of critical point is where $A'(x) = 0$. In fact, $A'(x) = 210 - 4x$, and so

$$0 = A'(x) = 210 - 4x \implies 4x = 210 \implies x = 52.5.$$

Now,

$$\begin{aligned} A(52.5) &= 52.5(210 - 2 \cdot 52.5) = 52.5(105) = 5512.5 \\ A(0) &= 0(210 - 0) = 0 \\ A(105) &= 105(210 - 2 \cdot 105) = 0. \end{aligned}$$

By the argument above, the absolute maximum of A must be one of these values. Clearly, the largest is $A(52.5) = 5512.5$. So the area of the largest possible holding pen is 5512.5m^2

6. (5 pts each)
   (a) Let $f(x) = x^2 + 2$. Use a left-hand sum with a uniform partition of four subintervals to estimate the area under the graph of f from $x = 1$ to $x = 3$.

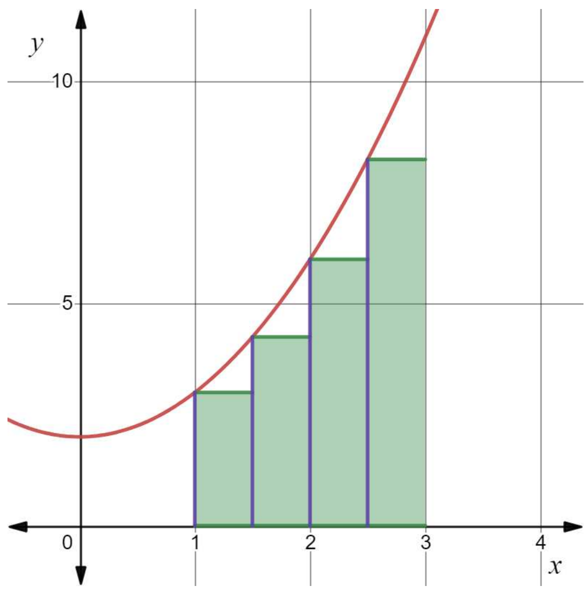
   Since we want to divide $[1, 3]$ into four uniform intervals, each interval will have width $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. The partition points are

   $$x_0 = 1, x_1 = \frac{3}{2}, x_2 = 2, x_3 = \frac{5}{2}, x_4 = 3.$$

The left-hand sum is

$$\begin{aligned}
L &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\
&= f(1)\frac{1}{2} + f\left(\frac{3}{2}\right)\frac{1}{2} + f(2)\frac{1}{2} + f\left(\frac{5}{2}\right)\frac{1}{2} \\
&= (1^2+2)\frac{1}{2} + \left(\left(\frac{3}{2}\right)^2+2\right)\frac{1}{2} + (2^2+2)\frac{1}{2} + \left(\left(\frac{5}{2}\right)^2+2\right)\frac{1}{2} \\
&= \frac{1}{2}\left(1+2+\frac{9}{4}+2+4+2+\frac{25}{4}+2\right) \\
&= \frac{43}{4}
\end{aligned}$$

(b) Sketch a diagram of your estimate. Do you think the left-hand sum you calculated in part (a) underestimates or overestimates the area under the graph of f between $x = 1$ and $x = 3$? Why?



It is clear from the diagram that the sum of the four rectangles is smaller than the area under the graph. This is because f is an increasing function, and so its value at the left endpoint of each subinterval is smaller than its values throughout the interval. Therefore the left-hand sum underestimates the area under the graph.

7. **Extra credit problem.** Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

(a) (8 pts) Determine if f is continuous at $x = 0$ by finding the limit

$$\lim_{x\to 0} f(x)$$

if it exists or showing that it does not exist. (Hint: You may want to look at one-sided limits and use the Squeeze Theorem or a $\delta - \epsilon$ argument.)

The key observation here is that if x is a number close to 0, then both x and x^2 are numbers that are close to 0. So it does not matter if x is rational or not, the value of $f(x)$ is close to 0 when x is close to 0. To make the argument more rigorous, let us first look at what happens if x is close to 0 and positive. If $0 < x < 1$ then we can multiply this inequality by x to get $0 < x^2 < x$. By LL7 and LL8,

$$\lim_{x\to 0^+} 0 = 0 \quad \text{and} \quad \lim_{x\to 0^+} x = 0.$$

By the Squeeze Theorem,

$$\lim_{x\to 0^+} f(x) = 0.$$

Let us see what happens when x is close to 0 and negative. If $-1 < x < 0$ then we can multiply this inequality by x to get $-x > x^2 > 0$. Note that the inequalities flipped because x is negative. By LL3 and LL8 (or by direct substitution into the polynomial),

$$\lim_{x\to 0^-} -x = -\lim_{x\to 0^-} x = 0.$$

By the Squeeze Theorem,

$$\lim_{x\to 0^-} f(x) = 0.$$

Since the left and the right limits are both 0,

$$\lim_{x\to 0} f(x) = 0.$$

Now, 0 is a rational number ($0 = 0/1$) and so $f(0) = 0$. Since

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

f is continuous at 0.

(b) (7 pts) Is f differentiable at 0? Why or why not?

The function f is differentiable at 0 if

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

exists. Since $f(0) = 0$,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}.$$

Note that

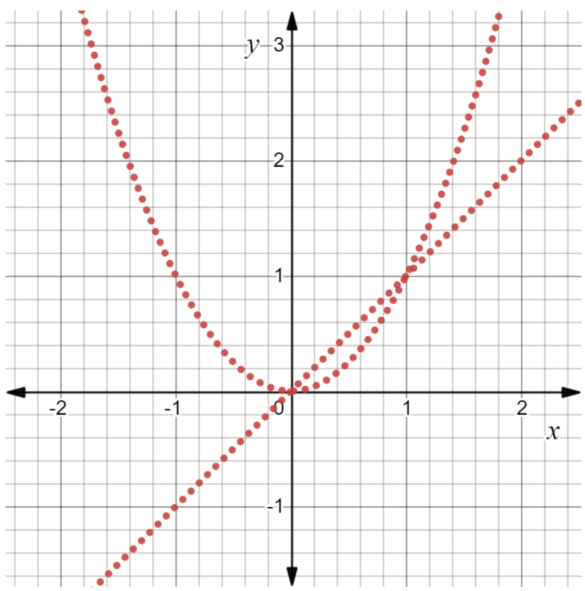
$$\frac{f(x)}{x} = \begin{cases} 1 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

That is if x is close to 0, then either $\frac{f(x)}{x} = 1$ or $\frac{f(x)}{x} = x$. Since the latter is close to 0, the value of $\frac{f(x)}{x}$ is either close to 1 or close to 0. In fact, no matter how small a neighborhood $(-\delta, \delta)$ of 0 we look at, there will be some values of x in there that are rational and others that are irrational. For example, if n is a large enough integer, then $1/n \in (-\delta, \delta)$ and $1/n$ is a rational number since it is a ratio of two integers. Similarly, if n is a large enough integer, then $\sqrt{2}/n \in (-\delta, \delta)$ and $\sqrt{2}/n$ is an irrational number. (The reason $\sqrt{2}/n$ is irrational is that if it could be expressed as ratio of two integers a/b, then $\sqrt{2}/n = a/b$ would imply $\sqrt{2} = na/b$, but $\sqrt{2}$ is irrational and cannot equal the ratio of the integers na and b.) In any case, no matter how close x has to be to 0, some of the values of $\frac{f(x)}{x}$ are close to 1 while others are close to 0, and therefore there is no number they are all getting arbitrarily close to. So

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

does not exist and hence f is not differentiable at 0.

If you are curious what the function f actually does, here is a description. The rational numbers are everywhere on the number line, and so are the irrational numbers. This means that no matter how small a piece of the number line you take, there are both rational and irrational numbers on it. Another way to think about this is that no matter where you stand on the number line, there are both rational and irrational numbers within your arm's reach, regardless of how small you are and how short your arms might be. This is actually what we argued in part (b). So the rational numbers on the number line look like a line with infinitely many holes punched in it, one hole at every irrational number. Vice versa, the irrational numbers on the number line look like a line with infinitely many holes punched in it, one hole at every rational number. Hence the graph of f looks like a combination of the line $y = x$ with infinitely many holes punched in it, one at every irrational value of x, and the parabola $y = x^2$ with infinitely many holes punched in it, one at every rational value of x. It is impossible to graph this on any physical device, but here is a very rough sketch to give you at least an idea of what we are talking about.



You can tell from this discussion that the graph of f is discontinuous at almost all values of $x = a$ because no matter how close to a you look, the values of f jump back and forth between values close to a and values close to a^2. There are two exceptions to this: $a = 0$ and $a = 1$ because $a = a^2$, at these numbers.