MCS 119 Exam 1 Solutions

1. (10 pts) Use the definition of continuity and the properties of limits to show that the function

$$f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}$$

is continuous at 2.

We need to show that

$$\lim_{x\to 2} f(x) = f(2).$$

Well,

$$\begin{aligned}
\lim_{x\to 2} f(x) &= \lim_{x\to 2}\left(3x^4 - 5x + \sqrt[3]{x^2+4}\right) \\
&= \lim_{x\to 2}(3x^4 - 5x) + \lim_{x\to 2}\sqrt[3]{x^2+4} && \text{by LL1} \\
&= \lim_{x\to 2}(3x^4 - 5x) + \sqrt[3]{\lim_{x\to 2}(x^2+4)} && \text{by LL11} \\
&= 3\cdot 2^4 - 5\cdot 2 + \lim_{x\to 2}\sqrt[3]{2^2+4} && \text{by direct substitution into the polynomials} \\
&= f(2).
\end{aligned}$$

2. (10 pts) Prove using the formal definition of the limit that

$$\lim_{x\to\infty} x^3 = \infty.$$

Hint: start by stating what the definition says when applied to this particular limit.

We need to show that for every M, there is a corresponding N such that if $x > N$ then $x^3 > M$. So, we want to know how large x needs to be to satisfy $x^3 > M$. Since the function $g(x) = \sqrt[3]{x}$ is increasing on all of $\mathbb{R}$,

$$x^3 > M \implies \sqrt[3]{x^3} > \sqrt[3]{M} \implies x > \sqrt[3]{M}.$$

This suggests that we shuold set $N = \sqrt[3]{M}$. Now, we will show that if $x > N$ then $x^3 > M$. Suppose

$$x > \sqrt[3]{M}.$$

Since the function $h(x) = x^3$ is increasing on all of $\mathbb{R}$,

$$x > \sqrt[3]{M} \implies x^3 > \left(\sqrt[3]{M}\right)^3 \implies x^3 > M.$$

3. Beavis and Butthead play frog baseball. As Beavis pitches the (poor, unlucky) frog, Butthead observes that the position of the frog measured as its distance in feet from Beavis is given by the function

$$f(t) = 90t - 13t^2$$

where t is time in seconds and $t = 0$ is the moment Beavis releases the frog.

(a) (6 pts) Use the definition of the derivative to find the instantaneous velocity of the frog at time $t = a$. What are the physical units of the derivative?

The instantaneous velocity at time $t = a$ is the derivative $f'(a)$:

$$\begin{aligned}
f'(a) &= \lim_{t\to a} \frac{f(t) - f(a)}{t - a} \\
&= \lim_{t\to a} \frac{90t - 13t^2 - (90a - 13a^2)}{t - a} \\
&= \lim_{t\to a} \frac{90t - 13t^2 - 90a + 13a^2}{t - a} \\
&= \lim_{t\to a} \frac{90t - 90a - 13t^2 + 13a^2}{t - a} \\
&= \lim_{t\to a} \frac{90(t - a) - 13(t^2 - a^2)}{t - a} \\
&= \lim_{t\to a} \frac{90(t - a) - 13(t - a)(t + a)}{t - a} \\
&= \lim_{t\to a}(90 - 13t - 13a) && \text{since } t \to a, \text{ we know } t \neq a, \text{ so } t - a \neq 0 \\
&= 90 - 13a - 13a && \text{by direct substitution int the polynomial} \\
&= 90 - 26a.
\end{aligned}$$

Since the units of f are feet and the units of t are seconds, the units of the difference quotient are ft/s. Therefore the units of f' are also ft/s.

(b) (4 pts) Use your result from part (a) to find the equation of the tangent line to f at $t = 3$.

First, the slope of that tangent line is

$$f'(3) = 90 - 26 \cdot 3 = 12,$$

and so its equation is $y = 12t + b$. The line passes through $(3, f(3)) = (3, 90\cdot 3 - 13\cdot 3^2) = (3, 153)$, and so

$$153 = 12 \cdot 3 + b \implies b = 153 - 36 = 117.$$

Hence the equation of the tangent line is $y = 12t + 117$.

4. (10 pts) Show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

We will show that the derivative of f does not exist at $x = 0$. By the definition of the derivative,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}.$$

Here are the left and the right limits:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{|x|}{x} &= \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1 \\
\lim_{x\to 0^-} \frac{|x|}{x} &= \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1
\end{aligned}$$

Since the one-sided limits are different, the two-sided limit does not exist. Hence $f'(0)$ does not exist.

5. (5 pts each)**Extra credit problem.** Let f be a function of real numbers. The *symmetric derivative* of f at x is defined as

$$f_s(x) = \lim_{h\to 0} \frac{f(x + h) - f(x - h)}{2h}.$$

The function f is *symmetrically differentiable* at x if the limit above exists.

(a) Prove that $f(x) = |x|$ is symmetrically differentiable at $x = 0$.

$$\begin{aligned} f_s(x) &= \lim_{h\to 0} \frac{f(0+h) - f(0-h)}{2h} \\ &= \lim_{h\to 0} \frac{|h| - |-h|}{2h} \\ &= \lim_{h\to 0} \frac{0}{2h} \\ &= 0 \end{aligned}$$

Since this limit exists, $f(x) = |x|$ is symmetrically differentiable at $x = 0$. Its symmetric derivative is 0 there.

Here is the geometric interpretation of what happens here. Notice that

$$\frac{f(0+h) - f(0-h)}{2h}$$

is the slope of the secant line between $x = -h$ and $x = h$. Because of the mirror symmetry of the graph about the y-axis, this secant line is horizontal for any value of h. This is why the symmetric derivative of $f(x) = |x|$ at $x = 0$ exists and is 0.

(b) Find a function $f : \mathbb{R} \to \mathbb{R}$ and a real number x such that f is not *symmetrically differentiable* at x. Make sure you justify your answer by showing that f_s does not exist.

There are of course many such functions. One is the piecewise defined function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

In this case,

$$\begin{aligned} f_s(x) &= \lim_{h\to 0} \frac{f(0+h) - f(0-h)}{2h} \\ &= \lim_{h\to 0} \frac{1 - 0}{2h} \\ &= \lim_{h\to 0} \frac{1}{2h} \end{aligned}$$

It is easy enought to see that this limit does not exist. If h is close to 0, then so is $2h$, and hence $\frac{1}{2h}$ is either a large positive number or a very negative number. In any case, as h gets closer and closer to 0, $\frac{1}{2h}$ does not get close to any particular real number.