MCS 119 Exam 2 Solutions

1. (10 pts) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist. Hint: note that you cannot do this by using the power rule because $\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3}$ is not a valid formula at $x = 0$, but you can use the definition of the derivative and show that the limit does not exist.

   By the definition of the derivative,
$$g'(0) = \lim_{x\to 0} \frac{x^{2/3} - 0^{2/3}}{x - 0} = \lim_{x\to 0} \frac{x^{2/3}}{x} = \lim_{x\to 0} \frac{1}{x^{1/3}} = \lim_{x\to 0} \frac{1}{\sqrt[3]{x}}$$
   As x gets close to 0 so does $\sqrt[3]{x}$. So depending on the sign of x, $1/\sqrt[3]{x}$ either gets large (and positive) or very negative. By this argument,
$$\lim_{x\to 0^+} \frac{1}{\sqrt[3]{x}} = \infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{\sqrt[3]{x}} = -\infty.$$
   That is neither the left nor the right limit exists. Therefore the two-sided limit does not exist either.

   If you want to be more rigorous about this, you can use exercise 1.4.62 from the homework. In this exercise, you proved that if $\lim_{x\to a} f(x)$ exists and is nonzero and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ cannot exist. Note that $\lim_{x\to 0} 1 = 1$ by LL7 and $\lim_{x\to 0} \sqrt[3]{x} = 0$ by LL10, and so $\lim_{x\to 0} \frac{1}{\sqrt[3]{x}}$ does not exist.

   A remark about an incorrect argument: You may be tempted to argue that
$$\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3} = \frac{2}{3}\frac{1}{\sqrt[3]{x}}$$
   which has a 0 in the denominator if $x = 0$ and therefore does not exist. Here is why this argument is not correct. That
$$\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3}$$
   is based on the following calculation. Let $g(x) = x^{2/3}$. Then
$$g'(a) = \lim_{x\to a} \frac{g(x) - g(a)}{x - a} = \lim_{x\to a} \frac{x^{2/3} - a^{2/3}}{x - a}.$$
   Using the standard identities $a^2 - b^2 = (a - b)(a + b)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, the numerator is
$$x^{2/3} - a^{2/3} = \sqrt[3]{x}^2 - \sqrt[3]{a}^2 = (\sqrt[3]{x} - \sqrt[3]{a})(\sqrt[3]{x} + \sqrt[3]{a}),$$
   and the denominator is
$$x - a = \sqrt[3]{x}^3 - \sqrt[3]{a}^3 = (\sqrt[3]{x} - \sqrt[3]{a})\big(\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{a} + \sqrt[3]{a}^2\big).$$
   We can cancel the common factor $\sqrt[3]{x} - \sqrt[3]{a}$:
$$\frac{x^{2/3} - a^{2/3}}{x - a} = \frac{\sqrt[3]{x} + \sqrt[3]{a}}{\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{a} + \sqrt[3]{a}^2}$$
   as long as $\sqrt[3]{x} - \sqrt[3]{a} \neq 0$, which we do know because $x \neq a$ as $x \to a$.
$$g'(a) = \lim_{x\to a} \frac{\sqrt[3]{x} + \sqrt[3]{a}}{\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{a} + \sqrt[3]{a}^2}$$
   Now,
$$\lim_{x\to a}(\sqrt[3]{x} + \sqrt[3]{a}) = \lim_{x\to a} \sqrt[3]{x} + \lim_{x\to a} \sqrt[3]{a} = \sqrt[3]{a} + \sqrt[3]{a} = 2\sqrt[3]{a}$$

by LL1, LL10, and LL7. Similarly,

$$\lim_{x\to a}\left(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2\right)=\lim_{x\to a}\sqrt[3]{x}^2+\lim_{x\to a}(\sqrt[3]{x}\sqrt[3]{a})+\lim_{x\to a}\sqrt[3]{a}^2=\sqrt[3]{a}+\sqrt[3]{a}=3\sqrt[3]{a}^2$$

by LL1, LL6, LL9, LL3, and LL7.

Now,

$$g'(a)=\lim_{x\to a}\frac{\sqrt[3]{x}+\sqrt[3]{a}}{\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2}=\frac{\lim_{x\to a}(\sqrt[3]{x}+\sqrt[3]{a})}{\lim_{x\to a}\left(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2\right)}$$

by LL5, if the limits in the numerator and the denominator both exist and the limit in the denominator is not 0. So if $a\neq 0$, then

$$g'(a)=\frac{2\sqrt[3]{a}}{3\sqrt[3]{a}^2}=\frac{2}{3}\frac{1}{\sqrt[3]{a}}.$$

This is what tells us that

$$\frac{d}{dx}x^{2/3}=\frac{2}{3}\frac{1}{\sqrt[3]{x}}$$

if $x\neq 0$.

And here is the catch: this is only true if $x\neq 0$. If $a=0$, then

$$\lim_{x\to a}\left(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2\right)=3\sqrt[3]{a}^2=0.$$

Therefore LL5 cannot be used and the above claculation that resulted in

$$g'(a)=\frac{2\sqrt[3]{a}}{3\sqrt[3]{a}^2}=\frac{2}{3}\frac{1}{\sqrt[3]{a}}$$

is not valid. That is

$$\frac{d}{dx}x^{2/3}=\frac{2}{3}\frac{1}{\sqrt[3]{x}}$$

is not a valid statement if $x=0$. So it cannot be used to conclude that the left-hand side does not exist because the right-hand side does not exist.

By the way, neither is it valid to argue that because LL5 cannot be used to evaluate $g'(a)$ if $a=0$, therefore $g'(0)$ does not exist. We have pointed out many times in class that the limit of a quotient of two fractions may exist even if LL5 cannot be used to calculate it because the limit of the denominator is 0.

2. (10 pts) Find equations of both lines through the point $(2,-3)$ that are tangent to the parabola $y=x^2+x$.

Since $y'(x)=2x+1$, the slope of the tangent line at some point a is $y'(a)=2a+1$. Hence the equation of the tangent line is $y=(2a+1)x+b$. We can find b by substituting the point (a,a^2+a) into the equation of the line:

$$a^2+a=(2a+1)a+b\implies b=a^2+a-(2a+1)a=-a^2.$$

So the equation of the tangent line at $x=a$ is $y=(2a+1)x-a^2$. This line passes through $(2,-3)$ if $(2,-3)$ satisfies the equation:

$$-3=(2a+1)2-a^2\iff a^2-4a-5=0\iff(a-5)(a+1)=0$$

So either $a=5$ or $a=-1$, and the equations of the two tangent lines that pass through $(2,-3)$ are

$$y=11x-25\qquad\text{and}\qquad y=-x-1.$$

3. (10 pts) Use the definition of the derivative to prove

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Hint: the trigonometric identity

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

may come in handy.

Here is how we did this in class:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0}\left[\sin(x)\,\frac{\cos(h)-1}{h}+\cos(x)\,\frac{\sin(h)}{h}\right] \\
&= \lim_{h\to 0}\left[\sin(x)\,\frac{\cos(h)-1}{h}\right]+\lim_{h\to 0}\left[\cos(x)\,\frac{\sin(h)}{h}\right] && \text{by LL1} \\
&= \sin(x)\,\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\,\lim_{h\to 0}\frac{\sin(h)}{h} && \text{by LL3}
\end{aligned}$$

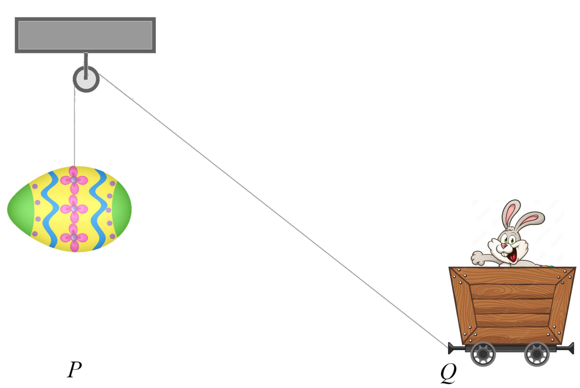
Since we know from Section 1.4 that

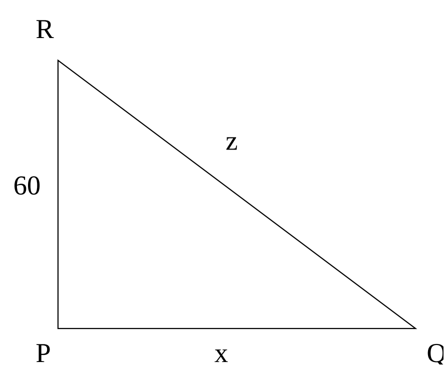
$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0,$$

we can now conclude

$$\frac{d}{dx}\sin(x) = \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x).$$

4. (10 pts) A popular ride in Easter Bunny Amusement Park consists of a cart tied to a 140-foot rope, which passes over a pulley at a height of 60 ft above the ground. The other end of the rope is tied to a humongous Easter egg. The weight of the egg pulls on the rope and moves the cart. When the cart is at a distance of 80 ft from the point directly below the pulley (point P in the diagram), the egg is falling toward the ground at a speed of 30 ft/s. How fast is the cart moving at the same time? The diagram below is definitely not drawn to scale.





Let R be the point where the pulley is. Then we have right triangle in the diagram above. By the Pythagorean Theorem, $z^2 = 60^2 + x^2$. Since x and z both change with time, they are functions of time t (which we will measure in seconds). Hence

$$\begin{aligned}\frac{d}{dt}z^2 &= \frac{d}{dt}(60^2 + x^2)\\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt}\end{aligned}$$

When $x = 80$, we can easily calculate $z = \sqrt{60^2 + 80^2} = 100$. We also know that $\frac{dz}{dt} = -30$ at this time, since the egg is pulling 30 ft or rope per second over the pulley. Hence

$$\frac{dx}{dt} = \frac{2z}{2x}\frac{dz}{dt} = \frac{100}{80}(-30) = -\frac{150}{4} = -37.5.$$

So the cart is moving toward the left at a speed of 37.5 ft$s$.

5. **Extra credit problem.** Let n be a positive integer. If n is odd, then let x be any real number; if n is even, then let x be any nonnegative real number. By the power rule,

$$\frac{d}{dx}\sqrt[n]{x} = \frac{d}{dx}x^{\frac{1}{n}} = \frac{1}{n}x^{1/n-1} = \frac{1}{n}\frac{1}{x^{\frac{n-1}{n}}}.$$

But we never proved in class that the power rule works for powers with fractional exponents. That is what you will do in this exercise. First, notice that if $y = \sqrt[n]{x}$ then $x = y^n$.

(a) (5 pts) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

We will differentiate $x = y^n$ with respect to x, keeping in mind that y is an implicit function of x. So

$$\begin{aligned}\frac{d}{dx}x &= \frac{d}{dx}y^n\\ 1 &= ny^{n-1}\frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{1}{ny^{n-1}}\end{aligned}$$

(b) (5 pts) Substitute $y = \sqrt[n]{x}$ into $\frac{dy}{dx}$ to express $\frac{dy}{dx}$ only in terms of x. Did you get the expected result?

$$\frac{dy}{dx} = \frac{1}{ny^{n-1}} = \frac{1}{n\sqrt[n]{x}^{n-1}} = \frac{1}{nx^{\frac{n-1}{n}}},$$

which is exactly what we wanted to show.