MCS 119 Final Exam Solutions

1. (a) (7 pts) Use the definition of the derivative to find the derivative of $f(x) = \sqrt{6-x}$. Be sure to explain every step of your calculation.

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\sqrt{6-x}-\sqrt{6-a}}{x-a} \\
&= \lim_{x\to a} \left[\frac{\sqrt{6-x}-\sqrt{6-a}}{x-a}\,\frac{\sqrt{6-x}+\sqrt{6-a}}{\sqrt{6-x}+\sqrt{6-a}}\right]
\end{aligned}
$$

Notice that we can multiply and divide by $\sqrt{6-x}+\sqrt{6-a}$ because it cannot be 0. Since $\sqrt{6-x} \geq 0$ and $\sqrt{6-a} \geq 0$, the only way $\sqrt{6-x}+\sqrt{6-a}$ could 0 is if both $\sqrt{6-x} = 0$ and $\sqrt{6-a} = 0$, which happens only if $x = a = 6$. But $x \to a$ means x is approaching a but $x \neq a$. Let us just work on the expression in the limit:

$$
\begin{aligned}
\frac{\sqrt{6-x}-\sqrt{6-a}}{x-a}\,\frac{\sqrt{6-x}+\sqrt{6-a}}{\sqrt{6-x}+\sqrt{6-a}} &= \frac{\sqrt{6-x}^2-\sqrt{6-a}^2}{(x-a)(\sqrt{6-x}+\sqrt{6-a})} \\
&= \frac{6-x-(6-a)}{(x-a)(\sqrt{6-x}+\sqrt{6-a})} \\
&= \frac{a-x}{(x-a)(\sqrt{6-x}+\sqrt{6-a})} \\
&= \frac{-(x-a)}{(x-a)(\sqrt{6-x}+\sqrt{6-a})}
\end{aligned}
$$

We can now cancel $x-a$ as it is not 0. Hence

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{-1}{\sqrt{6-x}+\sqrt{6-a}} \\
&= \frac{\lim_{x\to a}(-1)}{\lim_{x\to a}\left(\sqrt{6-x}+\sqrt{6-a}\right)} && \text{by LL5} \\
&= \frac{\lim_{x\to a}(-1)}{\lim_{x\to a}\sqrt{6-x}+\lim_{x\to a}\sqrt{6-a}} && \text{by LL1} \\
&= \frac{-1}{\sqrt{\lim_{x\to a}(6-x)}+\sqrt{6-a}} && \text{by LL7 and LL11} \\
&= \frac{-1}{\sqrt{6-a}+\sqrt{6-a}} && \text{by direct substitution} \\
&= -\frac{1}{2\sqrt{6-a}}
\end{aligned}
$$

So

$$f'(x) = \frac{-1}{2\sqrt{6-x}}.$$

Alternately, we could have argued that

$$\lim_{x\to a} \frac{-1}{\sqrt{6-x}+\sqrt{6-a}}$$

can be evaluated by direct substitution because the function $g(x) = \frac{-1}{\sqrt{6-x}+\sqrt{6-a}}$ is continuous. This is because we know $6-x$ is continuous, as it is a polynomial and so $\sqrt{6-x}$ is also continuous at every x in its domain by Theorem 6 in Section 1.5. Now, -1 and $\sqrt{6-a}$ do not depend on x, so they are constant functions and therefore continuous. Finally, sums and quotients of continuous functions are continuous by Theorem 4 in 1.6.

(b) (3 pts) What are the domains of f and f'? Do not forget to explain why.

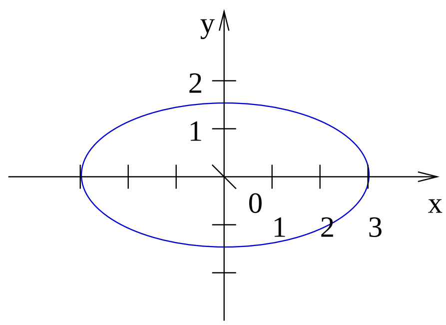
Since we cannot have a negative number under the square root in f, we need to have

$$6 - x \geq 0 \implies 6 \geq x.$$

For any other real number x, $f(x)$ has a real value. So $D(f) = (-\infty, 6]$.
Similarly, we cannot have a negative number under the square root in f', so we need $x \leq 6$. In fact, we do not want to have $x = 6$ either, otherwise the denominator is 0. For any other real number x, $f'(x)$ has a real value. So $D(f') = (-\infty, 6)$.

2. (10 pts) Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

First, here is a diagram:



We can find the slope of a tangent to the ellipse by implicit differentiation:

$$\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}36 \implies 2x + .8y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-2x}{8y} = -\frac{x}{4y}$$

If $y = mx + b$ is a line that passes through $(12, 3)$ and a point (x_0, y_0) on the ellipse then its slope is $m = \frac{y_0-3}{x_0-12}$. If this line is also tangent to the ellipse at (x_0, y_0), then its slope is also $m = -\frac{x_0}{4y_0}$. These two must be equal, so

$$-\frac{x_0}{4y_0} = \frac{y_0 - 3}{x_0 - 12}$$

We will multiply both sides by $x_0 - 12$ and by $4y_0$. But we need to be careful because if either of those is 0, this could result in false solutions. So we will need to check in the end to make sure we did not multiply by 0.

$$-x_0(x_0 - 12) = 4y_0(y_0 - 3)$$
$$-x_0^2 + 12x_0 = 4y_0^2 - 12y_0$$
$$0 = 4y_0^2 + x_0^2 - 12x_0 - 12y_0$$

Notice that $4y_0^2 + x_0^2 = 36$ since (x_0, y_0) is on the ellipse. Hence

$$0 = 36 - 12x_0 - 12y_0 \implies 0 = 3 - x_0 - y_0 \implies y_0 = 3 - x_0.$$

Substituting this into the equation of the ellipse gives

$$\begin{aligned}
x_0^2 + 4(3-x)^2 &= 36 \\
x_0^2 + 4(9 - 6x_0 + x_0^2) &= 36 \\
x_0^2 + 36 - 24x_0 + 4x_0^2 &= 36 \\
5x_0^2 - 24x_0 &= 0 \\
x_0(5x_0 - 24) &= 0
\end{aligned}$$

So $x_0 = 0$ or $x_0 = 24/5$. The corresponding values of y_0 are $y_0 = 3 - 0 = 3$ and $y_0 = 3 - 24/5 = -9/5$. Note that neither $x_0 - 12 = 0$ nor $4y_0 = 0$ at either of these two points, so we were justified multiplying by $x_0 - 12$ and $4y_0$ earlier. The derivative of the ellipse at $(0,3)$ is $y' = -0/((4)(3)) = 0$, and at $(24/5, -9/5)$, it is $y' = -\frac{24/5}{4(-9/5)} = \frac{2}{3}$. So the equations of the tangent lines are

$$y - 3 = 0(x - 0) \implies y - 3 = 0 \implies y = 3$$

$$y + \frac{9}{5} = \frac{2}{3}\left(x - \frac{24}{5}\right) \implies y = \frac{2}{3}x - \frac{16}{5} - \frac{9}{5} = \frac{2}{3}x - 5$$

3. (5 pts each)
   (a) Find an approximation to the integral

$$\int_0^4 (x^2 - 3x)\, dx$$

   using a Riemann sum with right endpoints and a uniform partition with 8 subintervals.

   Let $f(x) = x^2 - 3x$. The eight subintervals are

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right], \left[3, \frac{7}{2}\right], \left[\frac{7}{2}, 4\right].$$

   Each has width 1/2. Since we are using the right endpoints as sample points,

$$\begin{aligned}
R_8 &= \sum_{i=1}^{8} f(x_i)\frac{1}{2} \\
&= \frac{1}{2}\left[\underbrace{\left(\frac{1}{2}\right)^2 - 3\,\frac{1}{2}}_{f(x_1)} + \underbrace{1^2 - 3\cdot 1}_{f(x_2)} + \underbrace{\left(\frac{3}{2}\right)^2 - 3\,\frac{3}{2}}_{f(x_3)} + \cdots + \underbrace{4^2 - 3\cdot 4}_{f(x_8)}\right] \\
&= \frac{1}{2}\left[-\frac{5}{4} - 2 - \frac{9}{4} - 2 - \frac{5}{4} + 0 + \frac{7}{4} + 4\right] \\
&= -\frac{3}{2}
\end{aligned}$$

   (b) Use the Evaluation Theorem (or the Fundamental Theorem of Calculus) to evaluate

$$\int_0^4 (x^2 - 3x)\, dx.$$

$$\int_0^4 (x^2 - 3x)\, dx = \left[\frac{x^3}{3} - 3\,\frac{x^2}{2}\right]_0^4 = \frac{4^3}{3} - 3\,\frac{4^2}{2} - \frac{0^3}{3} + 3\,\frac{0^2}{2} = \frac{64}{3} - 24 = -\frac{8}{3}$$

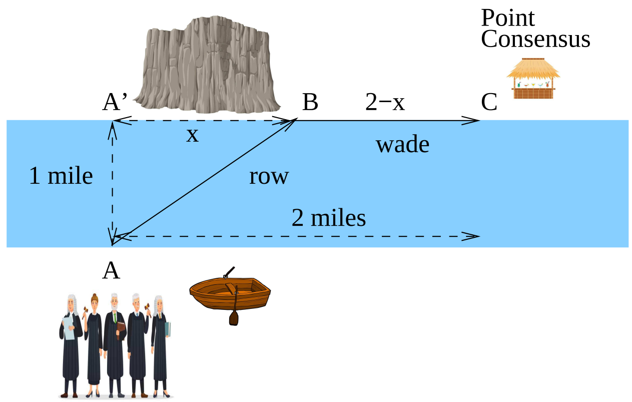
4. Let f be a function of real numbers and a a real number.
   (a) (3 pts) Define what it means for f to be differentiable at a.

   The function f is differentiable at a if the limit
   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
   exists.

   (b) (7 pts) Prove that if f is differentiable at a then f must be continuous at a.

   This is Theorem 4 in Section 2.2. Here is the proof we gave in class. Suppose f is differentiable at a, so the limit
   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
   exists. First, note that
   $$\lim_{x \to a} (x - a) = a - a = 0$$
   by direct substitution into the polynomial. Hence
   $$0 = 0 \cdot f'(a) = \left[\lim_{x \to a} (x - a)\right] \left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right].$$
   By LL4,
   $$\left[\lim_{x \to a} (x - a)\right] \left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right] = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} (x - a)\right].$$
   Note that we can cancel the factor $x - a$ since we know $x - a \neq 0$ as $x \to a$. This gives
   $$0 = \lim_{x \to a} [f(x) - f(a)].$$
   Hence
   $$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} [f(x) - f(a) + f(a)] \\ &= \lim_{x \to a} [f(x) - f(a)] + \lim_{x \to a} f(a) && \text{by LL1} \\ &= 0 + f(a) && \text{by LL7} \\ &= f(a) \end{aligned}$$
   This is exactly what we wanted to show.

5. (10 pts) The Row vs. Wade problem. As part of a team bulding exercise, the nine justices of the Supreme Court of the United States find themselves on the bank of a long and straight canal that is a mile wide. Their task is to reach Point Consensus (point C in the diagram), which is on the opposite bank of the canal, 2 miles from the point straight across from where they are standing. There is a boat they can row across the canal from point A to some point B on the other side. The boat is old and has a major leak (ha ha, pun intended), and therefore they can row it at a speed of only 2 miles/h. The opposite bank of the canal is a steep cliff the justices cannot climb. But the canal is shallow there and they can wade along the bank from point B to point C at 2.5miles/h. Find the optimal place of point B so the justices reach Point Consensus in the least amount of time by rowing across the canal and then wading along the bank. What is the minimal time for them to complete the trip? How do you know that what you found is the absolute minimum?



First, notice that this is the same problem as Example 4 in Section 3.5 with a slightly different story.

It is quite obvious that the optimal place for B should be somewhere between A' and C because it would not make it faster for the justices to land to the left of A' or to the right of C. Let x be the distance from A' to B in miles. Then we know $0 \le x \le 2$. By the Pythagorean Theorem, $AB = \sqrt{1+x^2}$. So it will take $\sqrt{1+x^2}/2$ hours to row from A to B. Then it will take $(2-x)/2.5$ hours to wade the remaining distance from B to C. So the total trip takes

$$f(x) = \frac{\sqrt{1+x^2}}{2} + \frac{2-x}{2.5} = \frac{\sqrt{1+x^2}}{2} + \frac{4}{5} - \frac{2}{5}x$$

hours, depending on x. The derivative

$$f'(x) = \frac{1}{2}\frac{1}{2}(1+x^2)^{-1/2}(2x) - \frac{2}{5}$$

for every $x \in \mathbb{R}$. So f is differentiable at every $x \in \mathbb{R}$. Hence f is continuous at every $x \in \mathbb{R}$. So we are looking for the absolute minimum of a contiunuous function over a closed interval. We can use the Closed Interval Method. Since f is differentiable, the only critical point is where

$$\begin{aligned}
0 &= f'(x) \\
&= \frac{x}{2\sqrt{1+x^2}} - \frac{2}{5} \\
\frac{x}{2\sqrt{1+x^2}} &= \frac{2}{5} \\
5x &= 4\sqrt{1+x^2} \\
25x^2 &= 16(1+x^2) \\
25x^2 &= 16 + 16x^2 \\
9x^2 &= 16 \\
x^2 &= \frac{16}{9} \\
x &= \pm\frac{4}{3}
\end{aligned}$$

Of these, only 4/3 is in $[0,2]$. Comparing

$$f(0) = \frac{\sqrt{1+0^2}}{2} + \frac{2-0}{2.5} = \frac{1}{2} + \frac{4}{5} = 1.3$$
$$f\left(\frac{4}{3}\right) = \frac{\sqrt{1+(4/3)^2}}{2} + \frac{2-4/3}{2.5} = \frac{\sqrt{25/9}}{2} + \frac{2/3}{5/2} = \frac{5}{6} + \frac{4}{15} = \frac{33}{30} = 1.1$$
$$f(2) = \frac{\sqrt{1+2^2}}{2} + \frac{2-2}{2.5} = \frac{\sqrt{5}}{2} \approx 1.12,$$

we can tell the absolute minimum of f is at 1.1 at $x = 4/3$. We know this is the absolute minimum because the absolute minimum of a continuous function on a closed interval must be either at one of the endpoints or must be a local minimum and hence a critical point, and we compared the value of f at the endpoints and at every critical point between them.

So the justices should row their boat to point B that is 4/3 of a mile from point A' toward point C. The least amount of time for them to get to point C is 1.1 hours.

6. (10 pts) In his relentless quest to catch the Road Runner, Wile E. Coyote is testing his ACME® rocket. The rocket is equipped with an accelerometer, which registers acceleration of $a(t) = 8+3t$, measured in m/s^2, t seconds after its launch. The rocket launches from rest at time $t=0$ and has enough fuel for 10 seconds. How far can Wiley fly in the 10 seconds before the rocket's fuel is exhausted?

Acceleration $a(t)$ is the derivative of velocity $v(t)$, which is the derivative of position $s(t)$. So v is one of the antiderivatives of a and s is one of the antiderivatives of v. The most general antiderivative of $a(t) = 8+3t$ is $v(t) = 8t + 3t^2/2 + c$. Since the rocket launches from rest, we know $0 = v(0) = c$. The most general antiderivative of $v(t) = 8t + 3/2t^2$ is $s(t) = 4t^2 + t^3/2 + d$. We have no information to find d, but we do not need to find d. We want to find Wiley's displacement between over the time interval $[0,10]$. That is

$$\Delta s = s(10) - s(0) = 4 \cdot 10^2 + \frac{10^3}{2} + d - \left(4 \cdot 0^2 + \frac{0^3}{2} + d\right) = 400 + 500 + d - d = 900.$$

So Wiley can fly 900 m before the rocket's fuel is exhausted.

7. **Extra credit problem.**

(a) (10 pts) Let f be a function or real numbers and $a \in \mathbb{R}$ such that

$$\lim_{x\to a} f(x) = \infty.$$

Prove that

$$\lim_{x\to a} \frac{1}{f(x)} = 0.$$

Hint: you will want to use the formal definition of the limit (as in $\delta - \epsilon$) and the infinite limit (as in $\delta - M$) for this argument.

That $\lim_{x\to a} f(x) = \infty$ tells us that for any positive M, there is some corresponding $\delta > 0$ such that if

$$0 < |x-a| < \delta$$

then

$$f(x) > M.$$

We want to show that for any every $\epsilon > 0$, there is some corresponding $\delta > 0$ such that if

$$0 < |x-a| < \delta$$

then

$$\left|\frac{1}{f(x)} - 0\right| < \epsilon.$$

So let $\epsilon > 0$. Then $M = 1\epsilon$ is a positive number. There must be a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$f(x) > M = \frac{1}{\epsilon}.$$

We will show that this δ also such that if

$$0 < |x - a| < \delta$$

then

$$\left|\frac{1}{f(x)} - 0\right| < \epsilon.$$

Since $M > 0$, it must also be true that $f(x) > 0$. We can now divide both sides of the inequality by M and by $f(x)$, and we get

$$M < f(x) \implies 1 < \frac{f(x)}{M} \implies \frac{1}{f(x)} < \frac{1}{M} = \epsilon.$$

Since $1/f(x)$ is positive,

$$\left|\frac{1}{f(x)} - 0\right| = \left|\frac{1}{f(x)}\right| = \frac{1}{f(x)} < \epsilon.$$

This proves

$$\lim_{x\to a} \frac{1}{f(x)} = 0.$$

(b) (5 pts) Is it also true that if

$$\lim_{x\to a} f(x) = 0$$

then

$$\lim_{x\to a} \frac{1}{f(x)} = \infty?$$

If you think it is true, prove it; if you do not think it it true, find a counterexample.

This is false. For example,

$$\lim_{x\to 0} x = 0$$

by LL7. But it is easy to see that

$$\lim_{x\to 0} \frac{1}{x} \neq \infty$$

because no matter how close x is to 0, it could be negative and then $1/x$ is negative and therefore cannot get arbitrarily large. In fact,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$