Notes for Section 1.5

Intuitively, a function is continuous if its graph is a continuous curve, that is it can be drawn without lifting up your pencil (pen, whatever writing utensil you use) from the paper (whiteboard, whatever you are writing on). But this intuitive notion would not make for a good formal definition because it depends on physical devices and their limitations and also on who is actually doing the drawing. In fact, there are continuous functions whose graphs would be physically quite impossible to draw with or without lifting up the drawing utensil. Therefore we need a more precise definition of what it means for a function to be continuous.

We saw in the previous section that polynomials and rational functions have the direct substitution property:

$$\lim_{x\to a} f(x) = f(a)$$

if f is a polynomial or a rational function whose domain includes a. This section is about what other kinds of functions have the direct substitution property and what functions do not. Actually, the official name of what we have been referring to as the direct substitution property is continuity.

Definition. Let f be a function of real numbers and $a \in \mathbb{R}$. We say f is *continuous* at a if

$$\lim_{x\to a} f(x) = f(a).$$

A function f that is not continuous at $a \in \mathbb{R}$ is called *discontinuous*. Note that $\lim_{x\to a} f(x)$ can only be equal to $f(a)$ if both the limit and $f(a)$ actually exist. In other words, continuity has three essential ingredients:

1. $f(a)$ must exist, that is a must be in the domain of f,
2. the limit $\lim_{x\to a} f(x)$ must exist,
3. the limit $\lim_{x\to a} f(x)$ must be equal to $f(a)$.

If any one of these three ingredients is missing, then f is discontinuous at a.

Notice that the definition of continuity we have here is a local property: it tells us what it means for a function to be continuous at a specific point, whereas the intuitive notion of a graph that can be drawn without lifting up the pencil is a global property. In fact, it may be hard to see how one is related to other at this point. This will become clearer if we look at how a function can fail to be continuous at a point.

Example 1 illustrates three ways for a function to be discontinuous at 1, 3, and 5. In fact, the three discontinuities in this example correspond exactly to what happens if one of the three ingredients of continuity is missing. Notice that if you want to trace the graph of the function in this example, you need to lift up your pencil at each of the three discontinuities.

Example 2 further illustrates different types of discontinuities with functions defined by formulas and also introduces their names. This is not to suggest that these are the only ways a function can fail to be continuous at a point, but these are common ways, which is why each has a name. Here are the precise definitions of these three kinds of discontinuities.

Definition. Let f be a function of real numbers and $a \in \mathbb{R}$. We say

1. f has a *removable discontinuity* at a if $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq f(a)$ (note that one reason for $\lim_{x\to a} f(x) \neq f(a)$ could be that $f(a)$ does not exist),
2. f has a *jump discontinuity* at a if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist but are not equal,
3. f has an *infinite discontinuity* at a if the value of $f(x)$ becomes arbitrarily large as x gets close to a either from the left or from the right, or both.

The names for jump and infinite discontinuity should make good sense. But why removable discontinuity? The idea there is that if f has a removable discontinuity at a then $L = \lim_{x\to a} f(x)$

must exist and therefore we can define a function g by

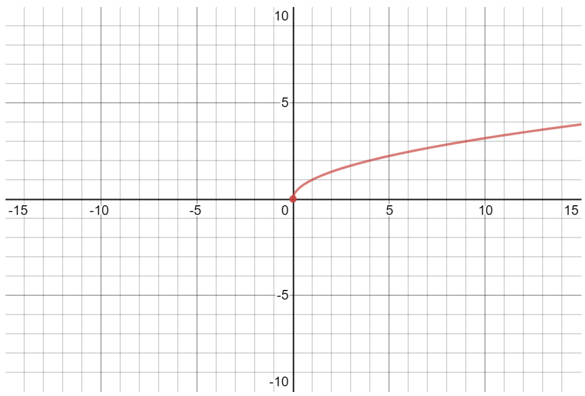
$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases}.$$

Now, g is closely related to f in that they have the same values at every x except at $x = a$. Therefore

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = L = g(a)$$

by one of the theorems we learned in Section 1.4 (see p. 38). Hence g is continuous at a. So we could easily remove the discontinuity of f by changing the behavior of f at only one point.

A function f that is continuous at a point a behaves well there in the sense that its value is what you would think it should be based on what number the value of f approaches as x gets close to a. Sometimes a function shows that good behavior on one side of a point a but not on the other. For example, the function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ does on the right of 0 but not on the left:



In such a case, we say f is right (or left continuous), which we can define much the same way we defined continuity except in terms of left and right limits. See Definition 2.

I pointed out already that continuity is a local property. The intuitive notion of continuity as having a graph that can be drawn without lifting up the pencil really corresponds to a lack of breaks or discontinuities in the graph. In other words, a function whose graph can be drawn without lifting up the pencil is one that is continuous at every point in its domain, or at least at every point of an interval over which the graph is drawn. This is the idea behind Definition 3 of the continuity over an interval. Note the special treatment of the endpoints if they are included in the interval.

As an example, consider the square root function $f(x) = \sqrt{x}$ over the interval $[0, \infty)$. It is continuous over this interval because if $a > 0$ then

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}$$

by LL10 and

$$\lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$$

also by LL10, only this time we applied LL10 to the right limit.

We define similarly what it means for a function to be continuous on its domain:

Definition. Let f be a function of real numbers. Then f is *continous on its domain* (or just continuous) if f is continuous at every $a \in D(f)$, except that if a is a left or right endpoint in the domain then only right/left continuity is required at a.

For example, we have just seen that $f(x) = \sqrt{x}$ is continuous on its natural domain $\mathbb{R}^{\geq 0}$.

Continuity is a very convenient property when it comes to evaluating limits, but how do we know which functions are continuous without having to evaluate limits? We can give an answer to this question that is similar to how introducing the limit laws helped us tackle many limits. A function that is contsructed from continuous ingredients using standard algebra also has to be continuous according to the following theorem, which is Theorem 4 in the textbook, except for the last part I added.

Theorem. *Let f and g be functions of real numbers that are both continuous at the real number a. Then the following functions are also continuous at a:*

1. $f + g$,
2. $f - g$,
3. cf *for any constant factor* $c \in \mathbb{R}$,
4. fg,
5. f/g *if* $g(a) \neq 0$,
6. f^n *for any positive integer* n.

Proof: These assertions easily follow from the corresponding limit laws. I will prove the first one and leave the rest to you since the arguments are basically the same.

To prove that $f + g$ is continuous at a, we need to show that

$$\lim_{x \to a} (f + g)(x) = (f + g)(a).$$

We can do it this way:

$$\begin{aligned} \lim_{x \to a} (f + g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) && \text{by LL1} \\ &= f(a) + g(a) && \text{since } f \text{ and } g \text{ are continuous at } a \\ &= (f + g)(a). \end{aligned}$$

□

Note that Theorem 4 is also true for one-sided limits.

Now we know how to build more complex continuous functions from simple continuous ingredients. But what are these simple continuous ingredients? We already know from Section 1.4 that polynomials and rational functions have the direct substitution property at every number a in their domains. We actually proved this in Section 1.4. This is restated in Section 1.5 as Theorem 5 using the language of continuity.

We also showed in Section 1.4 that the sine and the cosine functions have the direct substitution property at every real number a. The following theorem (Theorem 6 in the text, except I added the part about exponential and log functions) gives a more comprehensive list of continuous functions.

Theorem. *The following types of functions are continuous on their domains:*

1. *polynomials,*
2. *rational functions,*
3. *root functions,*
4. *trigonometric functions,*
5. *exponential and logarithmic functions.*

Proof: We already proved in Section 1.4 that polynomials have the direct substitution property at every $a \in R$, and rational functions also do at every $a \in \mathbb{R}$ that does not make the denominator 0.

That root functions $f(x) = \sqrt[n]{x}$ are continuous follows directly from LL10. Note that if n is even then 0 is the left endpoint of the domain $[0, \infty)$ and therefore only right continuity is needed there. This special case can be handled in exactly the same way we showed above that the square root function is continuous on $[0, \infty)$.

We showed in Section 1.4 that sine and cosine have the direct substitution property at every real number a, that is they are continuous at every $a \in \mathbb{R}$. That the tangent function is continuous at every real number in its domain (that is at every $a \neq \pi/2 + k\pi$ for $k \in \mathbb{Z}$) follows by noting that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and using Theorem 4 or just LL5. The real numbers in the domain of tangent are exactly those that do not make the denominator 0. Other trigonometric functions such as $\sec(x)$, $\csc(x)$, and $\cot(x)$ can be shown to be continuous on their domains by analogous arguments.

We do not have the necessary tools at this point to prove that exponential and logarithmic functions are continuous, which is probably why the textbook does not mention these. But I wanted you to know already that such functions are also continuous. □

One way to construct more complicated functions from simpler ones is composition. It turns out, composition also plays nicely with continuous functions: composing continuous functions results in a continuous function under appropriate conditions. Theorem 7 in the textbook is a somewhat more general result. Its proof is in Appendix C.

Theorem. *Let f and g be functions of real numbers and let a and b be real numbers such that*

$$\lim_{x \to a} g(x) = b$$

and f is continuous at b. Then

$$\lim_{x \to a} f(g(x)) = f(b).$$

Proof: The proof is relatively straightforward, but does require a $\delta - \epsilon$ argument. We want to show

$$\lim_{x \to a} f(g(x)) = f(b),$$

that is for any $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|f(g(x)) - f(b)| < \epsilon.$$

So let $\epsilon > 0$. Since f is continuous at b,

$$\lim_{y \to b} f(y) = f(b).$$

I used y instead of x here for a reason: we will need x later on. But do not let this throw you off. It does not matter what we call the input variable to f. What matters is that when this input variable gets closer and closer to b, the value of f at this input variable must get closer and closer to $f(b)$ because f is continuous at b. This means that there must exist some $\delta_1 > 0$ such that if

$$0 < |y - b| < \delta_1$$

then

$$|f(y) - f(b)| < \epsilon.$$

So if we make sure that the input to f is closer to b than a distance of δ_1, then we know that the value of f at that input is closer to $f(b)$ than ϵ. Remember that we usually require $0 < |y - b|$ so that we know $y \neq b$, because when $y = b$, the function f may not even have a value. But that is not a concern here. We know f is continuous at b, and therefore $f(b)$ exists. In fact, if $y = b$, then

$$|f(y) - f(b)| = |f(b) - f(b)| = |0| = 0,$$

which is certainly less than the positive number ϵ. Therefore it is enough for us to know that

$$|y - b| < \delta_1$$

to be able to conclude

$$|f(y) - f(b)| < \epsilon.$$

In the end, we want to show that $f(g(x))$ can be made to get arbitrarily close to $f(b)$. So it is $g(x)$ that we want within a distance of δ_1 to b. By substituting $g(x)$ for y in the above, we get that whenever the value oif $g(x)$ is such that

$$|g(x) - b| < \delta_1,$$

then

$$|f(g(x)) - f(b)| < \epsilon.$$

But we can in fact make the value of $|g(x) - b|$ as small as we need to because the limit of $g(x)$ is exactly b as $x \to a$. So there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|g(x) - b| < \delta_1.$$

This δ does exactly what we want. Suppose

$$0 < |x - a| < \delta.$$

Then

$$|g(x) - b| < \delta_1.$$

Therefore

$$|f(g(x)) - f(b)| < \epsilon.$$

That such a $\delta > 0$ exists is exactly what we needed to show. □

Here is an example how Theorem 7 could be used to calculate

$$\lim_{x \to 0} \sqrt{\frac{\sin(x)}{x}}.$$

We already know from Section 1.4 that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

We also know that $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$, and hence it is continuous at $x = 1$. By Theorem 7,

$$\lim_{x \to 0} \sqrt{\frac{\sin(x)}{x}} = \sqrt{1} = 1.$$

Theorem 8, stated below, is what we are really after. It is an immediate consequence of this last result. The proof in the textbook is short and simple enough, so I will not repeat it here.

Theorem. *Let f and g be functions of real numbers and let a such that g is continuous at a and f is continuous at $b = g(a)$. Then $f \circ g$ is continuous at a.*

Here is an example how Theorems 4, 6, and 8 can be used together to show that a function is continuous. Let

$$f(x) = \frac{1}{\sqrt{\sin^2(x) + 1}}.$$

We will argue that f is continuous at every $a \in \mathbb{R}$. First, note that $\sin(x)$ is continuous at every $a \in \mathbb{R}$ by Theorem 6. Also, the polynomial $g(x) = x^2 + 1$ is continuous at every real number by Theorem 6. So by Theorem 8, the composite function

$$g(\sin(x)) = \sin^2(x) + 1$$

is also continuous at every real number. Since $\sin^2(x) \geq 0$ and hence $\sin^2(x) + 1 \geq 1$, the number under the square root is always positive. By Theorem 6, the square root function is continuous at positive numbers, and so it is is continuous at $x \geq 1$. Hence the composite function

$$\sqrt{\sin^2(x) + 1}$$

is continuous at every $a \in \mathbb{R}$.

Now, the constant function $h(x) = 1$ is a polynomial and hence continuous at every $a \in \mathbb{R}$ by Theorem 6. Finally, since $\sin^2(x) + 1$ is always positive, $\sqrt{\sin^2(x) + 1} > 0$ and therefore the denominator is never 0. By Theorem 4, the quotient

$$f(x) = \frac{1}{\sqrt{\sin^2(x) + 1}}$$

is continuous at every $a \in \mathbb{R}$.

The last thing in this section on continuous functions is the Intermediate Value Theorem. The theorem is clearly stated in the book, so I will not repeat it here. What the theorem says should be intuitively clear if you think of continuous functions as ones whose graphs can be drawn without lifting up your pencil. Because you do not lift your pencil as you draw the graph between $(a, f(a)$ and $(b, f(b))$, the graph must cross the horizontal line $y = N$ at some point. This is illustrated in Figure 6. Of course, the graph could cross that horizontal line more than once, which is also illustrated in Figure 6. Even though what the theorem says makes good sense, the proof is beyond the level of our course. The reason the theorem holds has to do with how the real numbers fill the number line without leaving any gaps.

Note that the Intermediate Value Theorem (IVT from now on) gives no clue how to calculate a value c where $f(c) = N$. Thus it is more of a theoretical result than a practical one. It is used in proving other theorems rather than in solving computational problems.

The book gives one example (Example 8) for how the IVT can be used in a practical context. Here is another. We will show that the equation

$$\cos(x) = x$$

has a solution between 0 and $\pi/2$. Let

$$f(x) = \cos(x) - x$$

Note that f is continuous at every $x \in \mathbb{R}$ because $g(x) = \cos(x)$ and $h(x) = x$ are continuous by Theorem 6 (trig function and polynomial) and hence their difference $f(x) = g(x) - h(x)$ is continuous on all of $\mathbb{R}$ by Theorem 4. In particular, f is continuous on $[0, \pi/2]$. Now,

$$f(0) = \cos(0) - 0 = 1$$
$$f\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{2} < 0$$

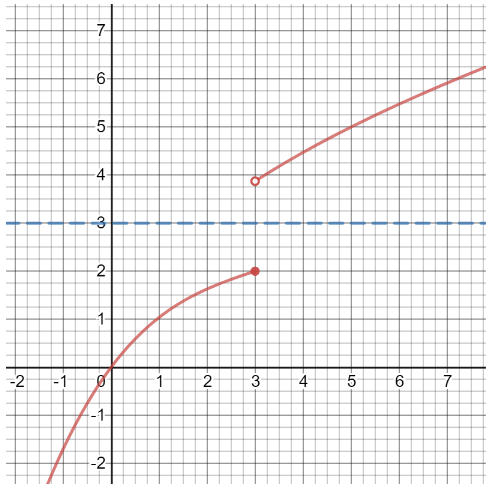
So $f(0) > 0 > f(\pi/2)$. By the IVT, there must exist a point $c \in (0, \pi/2)$ such that $f(c) = 0$. That is

$$\cos(c) - c = 0 \implies \cos(c) = c.$$

So the equation $\cos(x) = x$ has a solution c between 0 and $\pi/2$. How to find that solution is a different matter.

It is worth noting that continuity on the closed interval $[a, b]$ is an essential requirement for the function f in the IVT. If f were not continuous, it could easily jump over a value between $f(a)$

and $f(b)$. For an example, consider the function f whose graph is given below.



Notice that $f(0) = 0$ and $f(5) = 5$. Even though $f(0) < 3 < f(5)$, there is no point c in $[0, 5]$ such that $f(c) = 3$.