Notes for Section 1.6

This section starts off with an example we already looked at last semester. We found that

$$\lim_{x\to 0}\frac{1}{x^2}$$

does not exist because as the value of x gets close to 0, x^2 will be a small positive number and therefore $1/x^2$ will be large and will keep getting larger and larger as x gets closer to 0. So the value of $1/x^2$ does not get close to any particular real number. Although the limit does not exist, we can indicate the reason it does not exist and say something about the behavior of the function $f(x) = 1/x^2$ near 0 by saying

$$\lim_{x\to 0}\frac{1}{x^2} = \infty.$$

Here is what this means in somewhat informal terms.

Informal definition 1. Let f be a function of real numbers and let $a \in \mathbb{R}$. We say

$$\lim_{x\to a} f(x) = \infty$$

if the value of $f(x)$ can be made arbitrarily large (as large as you want) if x is made sufficiently close to a but not equal to a.

It is important to understand that this does not mean the limit of $f(x)$ exists as x approaches a. It does not mean that the value of $f(x)$ is getting close to ∞ either. In fact, there is no such thing as getting close to ∞. Wherever you are on the number line, ∞ is infinitely far away.

Here is a closer look at why $f(x) = 1/x^2$ satisfies this definition as $x \to 0$. We will show that we can make $f(x)$ as large as we want by making x close enough to 0. Let M be any real number. We will show we can make $f(x) > M$. While M could of course be negative or even 0, it would not be much of a challenge to make $f(x) > M$ in such a case. In fact, $1/x^2 > 0$ for any nonzero real number x. So we may as well assume that M is positive. No generality is lost that way. So, we want

$$\frac{1}{x^2} > M.$$

First, multiply both sides by x^2. Since $x \neq 0$ (because $x = 0$ is not even in the domain of f), we know that x^2 is positive. Therefore we do not need to worry about flipping the inequality when we multiply by x^2. We now have

$$1 > Mx^2.$$

We will now divide by M. Remember that we assumed $M > 0$, so we do not need to worry about dividing by 0 or flipping the inequality:

$$\frac{1}{M} > x^2.$$

Finally, we can take square roots of both side. This is because the square root function $g(x) = \sqrt{x}$ is an increasing function on its domain of $\mathbb{R}^{\geq 0}$. This means that if $0 \leq x^2 < 1/M$ then $g(x^2) < g(1/M)$, that is

$$\sqrt{x^2} < \sqrt{\frac{1}{M}}.$$

Since $\sqrt{x^2} = |x|$ and

$$\sqrt{\frac{1}{M}} = \frac{1}{\sqrt{M}},$$

we get

$$|x| < \frac{1}{\sqrt{M}}.$$

Note that it is once again important that we assumed $M > 0$ because we are taking the square root of it. This tells us that we want x to be close enough to 0 that

$$|x - 0| = |x| < \frac{1}{\sqrt{M}}.$$

We can now show that if x is close enough to 0 so that

$$|x - 0| < \frac{1}{\sqrt{M}}$$

that does in fact make

$$\frac{1}{x^2} > M$$

by verifying that the logic works backwards too. We start with

$$|x| < \frac{1}{\sqrt{M}}.$$

We can square both sides since the function $h(x) = x^2$ is increasing on $\mathbb{R}^{\geq 0}$ and we do know that $0 \leq |x| < 1/\sqrt{M}$, so $|x| < 1/\sqrt{M}$ implies

$$|x|^2 < \left(\frac{1}{\sqrt{M}}\right)^2 \implies x^2 < \frac{1}{M} \implies Mx^2 < 1 \implies M < \frac{1}{x^2}.$$

We can define

$$\lim_{x \to a} f(x) = -\infty$$

similarly.

Informal definition 2. Let f be a function of real numbers and let $a \in \mathbb{R}$. We say

$$\lim_{x \to a} f(x) = -\infty$$

if the value of $f(x)$ can be made arbitrarily negative (as negative as you want) if x is made sufficiently close to a but not equal to a.

An example of a negative infinite limit would be

$$\lim_{x \to 0} -\frac{1}{x^2} = -\infty.$$

Try to show that for any real number M, however negative, you can make

$$-\frac{1}{x^2} < M$$

by making x close enough to 0 via an argument similar to the argument above. You can start off by assuming $M < 0$. Be careful about the directions of the inequalities as you multiply/divide by negative numbers. I will leave the details to you.

One-sided limits

$$\lim_{x \to a^+} f(x) = \infty, \quad \lim_{x \to a^+} f(x) = -\infty, \quad \lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = -\infty$$

can be defined similarly. Try to write down the informal definitions. For examples of one-sided limits, look at Example 2 in the book.

Actually, we have seen such limits before: we called them infinite discontinuities. Another way to look at them is to notice that wherever such an infinite limit occurs, the graph of the function gets close to looking like a vertical line. There is a good chance you have seen such things in your prior studies and called them vertical asymptotes. We now have enough technical knowledge to define precisely what a vertical asymptote is.

Definition 1. f has an *vertical asymptote* at $x = a$ if at least one of

1. $\lim_{x\to a^+} f(x) = \infty$,
2. $\lim_{x\to a^+} f(x) = -\infty$,
3. $\lim_{x\to a^-} f(x) = \infty$,
4. $\lim_{x\to a^-} f(x) = -\infty$

is true.

Note that the book gives six conditions, but two of them are superfluous: if either

$$\lim_{x\to a} f(x) = \infty$$

or

$$\lim_{x\to a} f(x) = -\infty$$

then the left and the right limits are also ∞ or $-\infty$.

The book introduces you to limits at infinity via the example

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1.$$

Here is a somewhat less informal, although not entirely rigorous way to see this. When x is a large number, the value of x^2-1 is almost the same as the value of x^2. By almost the same, I mean that they only differ by 1, which is really small relative to the size of x^2. That is, if we approximate

$$x^2 - 1 \approx x^2,$$

the relative error (percentage error) we make is small. And it gets smaller and smaller as x gets larger. Similarly,

$$x^2 + 1 \approx x^2.$$

And hence

$$\frac{x^2-1}{x^2+1} \approx \frac{x^2}{x^2} = 1.$$

The crucial thing is that this approximation gets better and better as x gets larger and larger. So the value of $f(x) = \frac{x^2-1}{x^2+1}$ does in fact get closer and closer to 1 as x gets larger and larger.

The trick you have just learned works quite well in general. For another example, consider

$$\lim_{x\to\infty} \frac{3x^2+2}{2x^2+5x}.$$

When x is large

$$3x^2 + 2 \approx 3x^2$$

and

$$2x^2 + 5x \approx 2x^2.$$

In the second approximation, I used the idea that even though $5x$ is a large number, $2x^2$ is much larger. So

$$\frac{3x^2+2}{2x^2+5x} \approx \frac{3x^2}{2x^2} = \frac{3}{2}.$$

Hence

$$\lim_{x\to\infty} \frac{3x^2+2}{2x^2+5x} = \frac{3}{2}.$$

This kind of argument also works when $x \to -\infty$. When x is a negative number of large magnitude, x^2 is a really big number, and relative to x^2, 2 and $5x$ are much smaller in size. So

$$3x^2 + 2 \approx 3x^2 \quad \text{and} \quad 2x^2 + 5x \approx 2x^2.$$

Once again, these approximation get better and better as x gets more and more negative. So the approximation

$$\frac{3x^2+2}{2x^2+5x} \approx \frac{3x^2}{2x^2} = \frac{3}{2}$$

also keeps getting better and better as $x \to -\infty$ and hence

$$\lim_{x\to-\infty} \frac{3x^2+2}{2x^2+5x} = \frac{3}{2}.$$

Graph the function $f(x) = \frac{3x^2+2}{2x^2+5x}$ in Desmos to see that the graph is consistent with what we have just said.

The informal definitions of limits at ∞ and $-\infty$ are as follows.

Informal definition 3. Let f be a function of real numbers and let $L \in \mathbb{R}$. We say

$$\lim_{x\to\infty} f(x) = L$$

if the value of $f(x)$ can be made arbitrarily close (as close as you want) to L if x is made sufficiently large. Similarly, we say

$$\lim_{x\to-\infty} f(x) = L$$

if the value of $f(x)$ can be made arbitrarily close (as close as you want) to L if x is made sufficiently negative.

When either of these happens, the graph of f gets closer and closer to the horizontal line $y = L$ either as x heads off toward ∞ or as it heads off toward $-\infty$. Such a horizontal line is called a *horizontal asymptote* (see Definition 4 in the book). For example, the line $y = 3/2$ is a horizontal asymptote to the function $f(x) = \frac{3x^2+2}{2x^2+5x}$ above.

The function $f(x) = 1/x^n$ where n is any positive integer is an important example of a function with a horizontal asymptote. We will show that

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

for any $n \in \mathbb{Z}^+$. Actually, the argument we will give works just fine for any positive number n, not only for integers. We want to show that $1/x^n$ can be made arbitrarily close to 0 by making x sufficiently large. That is, if $\epsilon > 0$ is any number, however small, we can make

$$\left|\frac{1}{x^n} - 0\right| < \epsilon$$

for every value of x that is larger than a sufficiently large number N. So let $\epsilon > 0$. We want

$$\left|\frac{1}{x^n} - 0\right| < \epsilon.$$

Note that since $x \to \infty$ we can treat x as a positive number. Hence $1/x^n > 0$, and

$$\frac{1}{x^n} - 0 = \frac{1}{x^n} > 0 \implies \left|\frac{1}{x^n} - 0\right| = \frac{1}{x^n}.$$

We want

$$\frac{1}{x^n} < \epsilon.$$

First, multiply both sides by x^n. Since $x > 0$, we know that x^n is positive. Therefore we do not need to worry about flipping the inequality when we multiply by x^n. We now have

$$1 < \epsilon x^n.$$

We will now divide by ϵ. Remember that $\epsilon > 0$, so we do not need to worry about flipping the inequality:

$$\frac{1}{\epsilon} < x^n.$$

Finally, we can take n-th roots of both sides. This is because the n-th root function $g(x) = \sqrt[n]{x}$ is an increasing function on $\mathbb{R}^+$. This means that if $0 < 1/\epsilon < x^n$ then $g(1/\epsilon) < g(x^n)$, that is

$$\sqrt[n]{\frac{1}{\epsilon}} < \sqrt[n]{x^n}.$$

Since x is positive, $\sqrt[n]{x^n} = x$ and

$$\sqrt[n]{\frac{1}{\epsilon}} = \frac{1}{\sqrt[n]{\epsilon}},$$

we get

$$\frac{1}{\sqrt[n]{\epsilon}} < x.$$

This tells us that we want to set $N = 1/\sqrt[n]{\epsilon}$. As a reality check, note that the closer we want $1/x^n$ to be to 0, that is the smaller we make ϵ, the larger $N = 1/\sqrt[n]{\epsilon}$ will be. So x will have to be larger to be sufficiently large when ϵ is smaller. This makes sense.

We now need to show that if

$$x > N = \frac{1}{\sqrt[n]{\epsilon}}$$

then

$$\left|\frac{1}{x^n} - 0\right| < \epsilon.$$

So suppose

$$x > \frac{1}{\sqrt[n]{\epsilon}}.$$

Note that both sides are positive since ϵ is positive. We can raise both sides to the n-th power because the power function $h(x) = x^n$ is increasing on $\mathbb{R}^+$. Hence

$$x^n > \left(\frac{1}{\sqrt[n]{\epsilon}}\right)^n = \frac{1}{\epsilon}.$$

We now multiply both sides by ϵ, which is positive:

$$\epsilon x^n > 1,$$

and then divide both sides by x^n which is also positive:

$$\epsilon > \frac{1}{x^n}.$$

Finally, as we noted earlier,

$$\left|\frac{1}{x^n} - 0\right| = \frac{1}{x^n} < \epsilon.$$

This is what we wanted to show.

You can use a similar argument to show that

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

I will leave the details to you, but I want to caution you to be careful as you manipulate inequalities, take n-th roots, and raise things to the n-th power because some of the numbers are now negative.

Armed with this last result, we can revisit the example

$$\lim_{x\to\infty} \frac{3x^2+2}{2x^2+5x}$$

and give a more rigorous argument for why the limit is $3/2$. First, note that as long as $x^2 \neq 0$, we can divide the numerator and the denominator by it without changing the value of the fraction:

$$\frac{3x^2+2}{2x^2+5x} = \frac{\frac{3x^2+2}{x^2}}{\frac{2x^2+5x}{x^2}} = \frac{\frac{3x^2}{x^2}+\frac{2}{x^2}}{\frac{2x^2}{x^2}+\frac{5x}{x^2}} = \frac{3+\frac{2}{x^2}}{2+\frac{5}{x}}.$$

If $x \to \infty$, we can certainly treat x as nonzero, so

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^2+2}{2x^2+5x} &= \lim_{x\to\infty} \frac{3+\frac{2}{x^2}}{2+\frac{5}{x}} \\
&= \frac{\lim_{x\to\infty}\left(3+\frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(2+\frac{5}{x}\right)} && \text{by LL5} \\
&= \frac{\lim_{x\to\infty} 3 + 2\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2 + 5\lim_{x\to\infty}\frac{1}{x}} && \text{by LL1 and LL3} \\
&= \frac{3+2(0)}{2+5(0)} && \text{by LL7 and } \lim_{x\to\infty}\frac{1}{x^n} = 0 \\
&= \frac{3}{2}
\end{aligned}$$

Technically, the use of the Limit Laws is a bit problematic, since we know the limit laws work for ordinary limits but we do not know if they work for limits at infinity. But in fact, they can be verified to work for limits at infinity much the same way as for ordinary limits.

This is a trick that works well for finding limits of rational functions at infinity in general. The easiest way to do this is to divide by the largest power of x in the denominator. Here are two other examples.

We will find

$$\lim_{x\to\infty} \frac{x^2-2}{5x^3+2x-1}.$$

As before, we can divide the numerator and the denominator by x^3. Since $x \to \infty$, we do not need to be concerned about $x = 0$.

$$\frac{x^2-2}{5x^3+2x-1} = \frac{\frac{x^2-2}{x^3}}{\frac{5x^3+2x-1}{x^3}} = \frac{\frac{x^2}{x^3}-\frac{2}{x^3}}{\frac{5x^3}{x^3}+\frac{2x}{x^3}-\frac{1}{x^3}} = \frac{\frac{1}{x}-\frac{2}{x^3}}{5+\frac{2}{x^2}-\frac{1}{x^3}}.$$

Therefore

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2-2}{5x^3+2x-1} &= \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^3}}{5+\frac{2}{x^2}-\frac{1}{x^3}} \\
&= \frac{\lim_{x\to\infty}\left(\frac{1}{x}-\frac{2}{x^3}\right)}{\lim_{x\to\infty}\left(5+\frac{2}{x^2}-\frac{1}{x^3}\right)} && \text{by LL5} \\
&= \frac{\lim_{x\to\infty}\frac{1}{x} - 2\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 5 + 2\lim_{x\to\infty}\frac{1}{x^2} - \lim_{x\to\infty}\frac{1}{x^3}} && \text{by LL1 and LL3} \\
&= \frac{0-2(0)}{5+2(0)-1(0)} && \text{by LL7 and } \lim_{x\to\infty}\frac{1}{x^n} = 0 \\
&= 0
\end{aligned}$$

This trick even works when there are roots in the numerator or the denominator, that is the fraction is not a rational fraction. Here is one of those:

$$\lim_{x\to-\infty} \frac{3x-1}{\sqrt{x^2+3}+5}$$

We will divide the numerator and the denominator by x. Even though there is an x^2 under the square root, it actually behaves more like an x because of the square root. Another way to say this is that if we divided by x^2 under the square root, that x^2 is roughly the same as an x outside the square root. Since $x \to -\infty$, we do not need to be concerned about $x = 0$.

$$\frac{3x-1}{\sqrt{x^2+3}+5} = \frac{\frac{3x-1}{x}}{\frac{\sqrt{x^2+3}+5}{x}} = \frac{3-\frac{1}{x}}{\frac{\sqrt{x^2+3}}{x}+\frac{5}{x}}$$

Let us work now on the $\frac{\sqrt{x^2+3}}{x}$ part. First, remember that $\sqrt{x^2} = |x|$. Since $x \to -\infty$, we may assume that x is negative. Hence $|x| = -x$. That is $\sqrt{x^2} = -x$. Therefore $x = -\sqrt{x^2}$. (If any of this seems puzzling, try what happens when $x = -2$.) So

$$\frac{\sqrt{x^2+3}}{x} = \frac{\sqrt{x^2+3}}{-\sqrt{x^2}} = -\frac{\sqrt{x^2+3}}{\sqrt{x^2}} = -\sqrt{\frac{x^2+3}{x^2}} = -\sqrt{1+\frac{3}{x^2}}.$$

Hence

$$\begin{aligned}
\lim_{x\to-\infty} \frac{3x-1}{\sqrt{x^2+3}+5} &= \lim_{x\to-\infty} \frac{3-\frac{1}{x}}{-\sqrt{1+\frac{3}{x^2}}+\frac{5}{x}} \\
&= \frac{\lim_{x\to-\infty}\left(3-\frac{1}{x}\right)}{\lim_{x\to-\infty}\left(-\sqrt{1+\frac{3}{x^2}}+\frac{5}{x}\right)} && \text{by LL5} \\
&= \frac{\lim_{x\to-\infty} 3 - \lim_{x\to-\infty}\frac{1}{x}}{-\lim_{x\to-\infty}\sqrt{1+\frac{3}{x^2}}+5\lim_{x\to-\infty}\frac{1}{x}} && \text{by LL1 and LL3} \\
&= \frac{3-0}{-\sqrt{\lim_{x\to-\infty}\left(1+\frac{3}{x^2}\right)}+5(0)} && \text{by LL7, LL11 and } \lim_{x\to\infty}\frac{1}{x^n}=0 \\
&= \frac{3}{-\sqrt{\lim_{x\to-\infty} 1 + 3\lim_{x\to-\infty}\frac{1}{x^2}}} && \text{by LL1, LL3} \\
&= \frac{3}{-\sqrt{1+3(0)}} && \text{by LL7 and } \lim_{x\to\infty}\frac{1}{x^n}=0 \\
&= -3
\end{aligned}$$

Let us see what happens if the numerator has higher degree than then denominator. We will now find

$$\lim_{x\to\infty} \frac{5x^3+2x-1}{x^2-2}.$$

As before, we can divide the numerator and the denominator by x^2, and we do not need to be concerned about $x = 0$ since $x \to \infty$:

$$\frac{5x^3+2x-1}{x^2-2} = \frac{\frac{5x^3+2x-1}{x^2}}{\frac{x^2-2}{x^2}} = \frac{\frac{5x^3}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{2}{x^2}} = \frac{5x+\frac{2}{x}-\frac{1}{x^2}}{1-\frac{2}{x^2}}.$$

Therefore

$$\lim_{x\to\infty} \frac{5x^3+2x-1}{x^2-2} = \lim_{x\to\infty} \frac{5x+\frac{2}{x}-\frac{1}{x^2}}{1-\frac{2}{x^2}}.$$

Unlike before, we cannot use LL5 because the limit of numerator will turn out not to exist. But we can argue at least informally that as $x \to \infty$, the terms

$$\frac{2}{x}, \quad \frac{1}{x^2}, \quad \frac{2}{x^2}$$

all approach 0. The $5x$ term gets larger and larger, while the 1 just stays at 1. So the numerator keeps getting large, arbitrarily large, while the denominator gets closer and closer to 1. A large number divided by a number that is close to 1 is still large. In fact, we can make the value of the fraction as large as we want by making $5x$ larger and larger until $5x + \frac{2}{x} - \frac{1}{x^2}$ is large enough that divided by $1 - \frac{2}{x^2}$ (which is actually a little less than 1), the result is as large as we wished. Hence the limit is ∞. The last part of this argument is definitely quite informal. In fact, we have not even defined (informally or formally) what

$$\lim_{x\to\infty} f(x) = \infty$$

means. Let me do this in a bit. Your textbook does not get around to it until Example 9 either.

Example 6 in the textbook uses a similar argument to this last one, combined with the trick of multiplying and dividing by the conjugate, which we have used in the past to deal with limits of functions that had square roots in them. The book ends up concluding that

$$\lim_{x\to\infty} (\sqrt{x^2+1} - x) = 0$$

by arguing quite informally. Here is how you could do the same rigorously. Let us start at the beginning. First, note that

$$\sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x}$$

because multiplying and dividing by the same nonzero number does not change the value. Wait, how do we know $\sqrt{x^2+1} + x \neq 0$? Since $x \to \infty$, we can certainly think of x as a positive number. Therefore x^2+1 is also positive (in fact, it is more than 1) and so $\sqrt{x^2+1}$ is positive as well. Hence $\sqrt{x^2+1} + x$ is the sum of two positive numbers, and so it definitely cannot be 0. Now,

$$\sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \frac{\sqrt{x^2+1}^2 - x^2}{\sqrt{x^2+1} + x} = \frac{x^2+1-x^2}{\sqrt{x^2+1} + x} = \frac{1}{\sqrt{x^2+1} + x}.$$

Therefore

$$\lim_{x\to\infty} (\sqrt{x^2+1} - x) = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x}.$$

I will show you two ways to evaluate this limit. For one, we can use the same trick we have seen earlier: we will divide both the numerator and the denominator by x, since the x^2 under the square root behaves much the same as an x. Since $x \to \infty$, we still do not need to be concerned that x could be 0, and so dividing by x is allowed.

$$\frac{1}{\sqrt{x^2+1} + x} = \frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}+x}{x}} = \frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}}{x} + 1}.$$

Since x is positive, $\sqrt{x^2} = |x| = x$. Hence

$$\frac{\sqrt{x^2+1}}{x} = \frac{\sqrt{x^2+1}}{\sqrt{x^2}} = \sqrt{\frac{x^2+1}{x^2}} = \sqrt{1 + \frac{1}{x^2}}.$$

Therefore

$$\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+1}-x) &= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} \\
&= \lim_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1} \\
&= \frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}\left(\sqrt{1+\frac{1}{x^2}}+1\right)} && \text{by LL5} \\
&= \frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}}+\lim_{x\to\infty}1} && \text{by LL1} \\
&= \frac{\lim_{x\to\infty}\frac{1}{x}}{\sqrt{\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)}+1} && \text{by LL11} \\
&= \frac{0}{\sqrt{1+0}+1} && \text{by LL1, LL7 and } \lim_{x\to\infty}\frac{1}{x^n}=0 \\
&= \frac{0}{2} \\
&= 0.
\end{aligned}$$

Here is another way to conclude the same thing. Note that

$$x^2+1>x^2 \implies \sqrt{x^2+1}>\sqrt{x^2}$$

because x^2 and x^2+1 are both nonnegative numbers, and $g(x)=\sqrt{x}$ is increasing on $\mathbb{R}^{\geq 0}$. So

$$x^2+1>x^2 \implies g(x^2+1)>g(x^2).$$

We do know that x is positive, and so $\sqrt{x^2}=x$. Hence

$$\sqrt{x^2+1}>x \implies \sqrt{x^2+1}+x>x+x=2x.$$

Since $2x$ and $\sqrt{x^2+1}+x$ are both positive numbers, and we have just seen that $\sqrt{x^2+1}+x$ is the larger of them, it follows that

$$0\leq\frac{1}{\sqrt{x^2+1}+x}\leq\frac{1}{2x}.$$

Now, notice that

$$\lim_{x\to\infty}\frac{1}{2x}=\frac{1}{2}\lim_{x\to\infty}\frac{1}{x}=\frac{1}{2}\cdot 0=0.$$

Of course, it is also true that

$$\lim_{x\to\infty}0=0.$$

It now follows by the Squeeze Theorem (which does work for limits at infinity) that

$$\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0.$$

Example 7 introduces you to another useful trick. It makes intuitive sense, but is a bit informal. But the argument can be made rigorous quite easily. Note that we already know

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

Since the sine function is continuous at every real number and in particular, at 0, we can use Theorem 7 from Section 1.5:

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \sin\left(\lim_{x\to\infty} \frac{1}{x}\right) = \sin(0) = 0.$$

We have argued a little earlier that

$$\lim_{x\to\infty} \frac{5x^3 + 2x - 1}{x^2 - 2} = \infty.$$

But in fact, we have not yet defined even informally what this means. If you read through the argument we gave, the informal definition is in there.

Informal definition 4. Let f be a function of real numbers. We say

$$\lim_{x\to\infty} f(x) = \infty$$

if the value of $f(x)$ can be made arbitrarily large (as large as you want) if x is made sufficiently large.

We can define

$$\lim_{x\to-\infty} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = -\infty, \quad \text{and} \lim_{x\to-\infty} f(x) = -\infty$$

similarly. Try writing down the definitions for yourself.

It is time to give formal defininitions of infinite limits and limits at infinity. The definitions are quite similar to the formal definition of the limit in terms of $\delta - \epsilon$. The ingredients differ slightly, but in predictable ways. I will not list all of these because there are so many of them. But the same phrases keep recurring in them all, and so you should be able to construct the missing ones. I will start with infinite limits.

Definition 2. Let f be a function of real numbers and let $a \in \mathbb{R}$. We say

$$\lim_{x\to a} f(x) = \infty$$

if for every number $M > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$f(x) > M.$$

Strictly speaking, it is not necessary to require that M be positive, but in practice, it is often helpful to know that when using the definition. As I argued at the very beginning, no generality is lost if we do. If we can make $f(x)$ larger than any positive number, then we can also make it larger than any real number. You have already seen an example of how to use this definition at the very beginning when we argued (completely rigorously) that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

The textbook does the exact same thing in Example 12.

We can define the negative infinite limit and one-sided infinite limits similarly.

Definition 3. Let f be a function of real numbers and let $a \in \mathbb{R}$. We say

$$\lim_{x\to a} f(x) = -\infty$$

if for every number $M < 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$f(x) < M.$$

Definition 4. Let f be a function of real numbers and let $a \in \mathbb{R}$. We say

$$\lim_{x \to a^+} f(x) = \infty$$

if for every number $M > 0$ there is a corresponding $\delta > 0$ such that if

$$a < x < a + \delta$$

then

$$f(x) > M.$$

I will leave the definitions of the remaining three one-sided limits to you to spell out.

For an example of how to use the definition of a one-sided limit, we will show that

$$\lim_{x \to 3^-} \frac{1}{x-3} = -\infty.$$

We want to show that for any $M < 0$ there is a corresponding $\delta > 0$ such that if

$$3 - \delta < x < 3$$

then

$$\frac{1}{x-3} < M.$$

So let $M < 0$. We want

$$\frac{1}{x-3} < M.$$

Multiply both sides by $x - 3$. Keep in mind that $x \to 3^-$, so $x < 3$. Therefore $x - 3 < 0$, and when we multiply by $x - 3$, the inequality flips:

$$1 > M(x-3).$$

We now divide by M. We know M is negative, so the inequlity flips again:

$$\frac{1}{M} < x - 3.$$

Note that this is one of those instances when it is convenient to know that M is negative, so we do not need to consider several cases and we do not need to worry about what happens if $M = 0$. This is why it is good to include $M < 0$ in the definition. We now add 3 to both sides:

$$3 + \frac{1}{M} < x.$$

Remember that M is a negative number, probably of quite large size, so $1/M$ is a small negative number, and hence $3 + 1/M$ is a little less than 3. Since this is a left limit, we also want $x < 3$. So x needs to be between 3 and the number $3 + 1/M$, which is a little less than 3. Therefore the choice we want to make for δ is that

$$3 - \delta = 3 + \frac{1}{M} \implies \delta = -\frac{1}{M}.$$

As a quick reality check, remember that $1/M$ is a small negative number, so δ is a small positive number, exactly as it should be. We now need to show that if

$$3 - \delta < x < 3$$

then

$$\frac{1}{x-3} < M.$$

So suppose

$$3 - \delta < x < 3.$$

Then

$$x > 3 - \delta = 3 + \frac{1}{M}.$$

Now, subtract 3, then multiply by M (which is negative), and finally divide by $x - 3$ (which is also negative because $x < 3$:

$$x > 3 + \frac{1}{M} \implies x - 3 > \frac{1}{M} \implies M(x-3) < 1 \implies M > \frac{1}{x-3},$$

which is what we wanted to show.

Here are the formal definitions of the limit at ∞ and at $-\infty$

Definition 5. Let f be a function of real numbers and let $L \in \mathbb{R}$. We say

$$\lim_{x \to \infty} f(x) = L$$

if for every $\epsilon > 0$ there is a corresponding $N > 0$ such that if

$$x > N$$

then

$$|f(x) - L| < \epsilon.$$

Similarly, we say

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\epsilon > 0$ there is a corresponding $N < 0$ such that if

$$x < N$$

then

$$|f(x) - L| < \epsilon.$$

You have already seen how to use this definition when we proved that

$$\lim_{x \to \infty} \frac{1}{x^n} = 0.$$

Finally, there are four formal definitions of infinite limits at infinity. I will spell out one of them and will leave the other three for you to write out.

Definition 6. Let f be a function of real numbers. We say

$$\lim_{x \to \infty} f(x) = \infty$$

if for every $M > 0$ there is a corresponding $N > 0$ such that if

$$x > N$$

then

$$f(x) > M.$$

As an example, we will show that

$$\lim_{x \to \infty} x^4 = \infty.$$

We want to show that for every $M > 0$ there is a corresponding $N > 0$ such that if

$$x > N$$

then

$$x^4 > M.$$

So let $M > 0$. We want

$$x^4 > M.$$

We can take 4th roots of both sides because x^4 and M are both positive and the 4th root function $g(x) = \sqrt[4]{x}$ is increasing on its domain $\mathbb{R}^{\geq 0}$. Hence

$$x^4 > M \implies \sqrt[4]{x^4} > \sqrt[4]{M}.$$

Note that $\sqrt[4]{x^4} = |x|$, and since x is positive as $x \to \infty$, $\sqrt[4]{x^4} = x$. So we need to have

$$x > \sqrt[4]{M}.$$

This tells us we want to set $N = \sqrt[4]{M}$. We now need to show that if

$$x > N$$

then

$$f(x) > M.$$

Suppose

$$x > N = \sqrt[4]{M}.$$

We can raise both sides to the 4th power because the function $h(x) = x^4$ is increasing on $\mathbb{R}^{\geq 0}$ and both x and $\sqrt[4]{M}$ are nonnegative. So

$$x > \sqrt[4]{M} \implies x^4 > \sqrt[4]{M}^4 = M,$$

which is what we wanted to show.