Notes for Section 2.1

We will start using what we have learned about limits to study the instantaneous rate of change of a function. Recall (and reread?) the motivating example and the discussion of average velocity/speed at the beginning of the notes for Section 1.3. Oddly, the book does not start Section 2.1 with this reference back to what we already learned in Section 1.3, but it does revisit it on p. 77. It seems more logical to me to start this section there. We concluded at that time that for an object moving along a straight line whose position at time t is given by the function $f(t)$, the average velocity of that object between t_1 and t_2 is

$$v_{\text{ave}} = \frac{\Delta f}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

Remember that the ratio above is called the *difference quotient*. We also noted at the time that if we are interested in the instantaneous velocity/speed then we need to find what number the difference quotient $\frac{\Delta f}{\Delta t}$ approaches as Δt gets closer and closer to 0. We did not have the concept of limit yet to formulate this idea precisely. But we do now, and we are ready to say that the instantaneous velocity of the moving object a time t_0

$$v_{\text{inst}} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

Note that the average velocity is always between two points in time and the instantaneous velocity is at one specific point in time.

Example 3 revisits the example of the ball dropped from the CN Tower in Toronto. We looked at this back in Section 1.3. The ball falls toward the ground along a straight line and its position is given by the function $f(t) = 4.9t^2$, where t is in seconds and $f(t)$ measures the distance the ball has fallen from its initial position in meters. We now want to find the instantaneous velocity of the ball after 5 seconds and also at the moment it hits the ground. I will first do these calculations a little differently from how the book does them. By the formula we gave above, the instantaneous velocity at $t_0 = 5$ is

$$\begin{aligned} v_{\text{inst}} &= \lim_{t \to 5} \frac{f(t) - f(5)}{t - 5} \\ &= \lim_{t \to 5} \frac{4.9t^2 - 4.9(5^2)}{t - 5} \end{aligned}$$

The function in the limit is a rational function, but we cannot find the limit by direct substitution because that would make the denominator (and the numerator for that matter) 0. In general, you cannot find the limit of the difference quotient as $\Delta t \to 0$ for exactly this reason. But we still know how to find such a limit. First, we will factor out the 4.9 and use LL3 to move the factor 4.9 in front of the limit:

$$v_{\text{inst}} = \lim_{t \to 5} \frac{4.9(t^2 - 5^2)}{t - 5} = 4.9 \lim_{t \to 5} \frac{t^2 - 5^2}{t - 5}.$$

Now, we note that $t^2 - 5^2 = (t+5)(t-5)$ and so

$$\frac{t^2 - 5^2}{t - 5} = \frac{(t+5)(t-5)}{t - 5} = t + 5$$

as long as $t - 5 \neq 0$. But we know $t - 5 \neq 0$ because $t \neq 5$ as $t \to 5$. So

$$\begin{aligned} v_{\text{inst}} &= 4.9 \lim_{t \to 5} (t + 5) \\ &= 4.9(5 + 5) && \text{by direct substitution into the polynomial} \\ &= 49 \, \frac{\text{m}}{\text{s}}. \end{aligned}$$

The reason the unit is m/s is that $f(t)$ and hence $f(t) - f(5)$ is in meters, while t and hence $t - 5$ is in seconds. Taking the limit as $t \to 5$ does not change the units of the difference quotient and the limit is the quantity the difference quotient approaches, so the limit must be measured in the same units.

To find the instantaneous velocity at the time the ball hits the ground, we first note that since the ball is dropped from a height of 450 m, it reaches the ground when t satisfies $450 = 4.9t^2$. That is

$$450 = 4.9t^2 \implies t^2 = \frac{450}{4.9} \implies t = \sqrt{\frac{450}{4.9}} \approx 9.6\,\mathrm{s}.$$

We could now redo the above calculation with 9.6 instead of 5. Or we could just find the instantaneous velocity at time $t_0 = a$ in general, and then substitute $a = \sqrt{\frac{450}{4.9}}$. The latter is actually simpler since we do not have to deal with messy numbers while finding the limit.

$$\begin{aligned}
v_{\text{inst}} &= \lim_{t\to a} \frac{f(t) - f(a)}{t - a} \\
&= \lim_{t\to a} \frac{4.9t^2 - 4.9a^2}{t - a} \\
&= 4.9 \lim_{t\to a} \frac{t^2 - a^2}{t - a} && \text{by LL3} \\
&= 4.9 \lim_{t\to a} \frac{(t+a)(t-a)}{t - a} && \text{we cancel } t - a \text{ because } t \neq a \text{ as } t \to a \\
&= 4.9 \lim_{t\to a} (t + a) \\
&= 4.9(a + a) && \text{by direct substitution into the polynomial} \\
&= 9.8a.
\end{aligned}$$

At $a = \sqrt{\frac{450}{4.9}}$,

$$v_{\text{inst}} = 9.8\sqrt{\frac{450}{4.9}} \approx 94\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The book uses a slightly different formula to do these calculations. Let me explain where that formula comes from. In

$$v_{\text{inst}} = \lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0},$$

we let $\Delta t = t - t_0$. Then $t = t_0 + \Delta t$. Now, instead of looking at the limit as $t \to t_0$, we look at what happens as $\Delta t \to 0$. So

$$v_{\text{inst}} = \lim_{\Delta t\to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}.$$

To make formulas a little simpler, it is common practice to use the letter h instead of Δt, as in

$$v_{\text{inst}} = \lim_{h\to 0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

Now, if we wanted to find the instantaneous velocity at $t_0 = 5$, we would find

$$v_{\text{inst}} = \lim_{h\to 0} \frac{f(5 + h) - f(5)}{h}.$$

Or to at $t_0 = a$, we would calculate

$$v_{\text{inst}} = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}.$$

The book shows you how to do the latter. And the former is almost the same with 5 instead of a.

These ideas are not only good for finding the speed/velocity of a moving object. They work for how fast the value of any function $f(x)$ changes as the value of its input x changes. That is called the instantaneous rate of change of f. Such rates are all over the place in the natural and social sciences and are important not only for calculating how quickly a function's value changes, but also for understanding conceptual relationships between varying quantities in response to changes in one another. The input variable does not need to be time. The function $f(x)$ could be measuring the concentration of one of the products of a chemical reaction as the function of the concentration of one of the reactants. Or $f(x)$ could be the population of a predator as a function of the population of its prey. Even your personal income tax rate is an example of a rate. If you are in the 12% federal tax bracket, that means your income tax liability to Uncle Sam changes (increases) by \$0.12 for every additional dollar you earn this year. In fact, we can generalize the idea of average velocity to any function $f(x)$.

Definition. Let f be a function of real numbers and let $x_1, x_2 \in \mathbb{R}$. The *average rate of change of f at between x_1 and x_2* is

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

The average rate of change is sometimes denoted $\frac{\Delta f}{\Delta x}$ since y is typically the name of the vertical axis when a function $f(x)$ is graphed. Remember that we called $\frac{f(x_1)-f(x_2)}{x_1-x_2}$ the *difference quotient.*

Similarly, we can generalize the idea of instantaneous velocity to any function $f(x)$.

Definition. Let f be a function of real numbers and let $a \in \mathbb{R}$. The *derivative of f at a* or the *instantaneous rate of change of f at a* is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Alternately, $f'(a)$ is also

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Both of these formulas are useful. Sometimes one is easier to use than the other in a calculation. Sometimes they are about the same. You need to learn how to use both of them. The quotients

$$\frac{f(x) - f(a)}{x - a} \quad \text{and} \quad \frac{f(a+h) - f(a)}{h}$$

can both be called the difference quotient. Another notation of the derivative of f at a is

$$\left.\frac{df}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{df}{dx}\right]_{x=a}.$$

These may look awkward compared to the simple $f'(a)$ for now, but there is a reason why people use this notation as well, and you will find out a little later in the course. The notation comes from the fact that

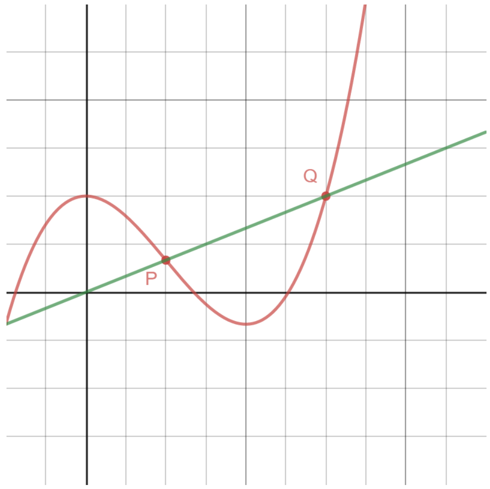
$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

The df and the dx are to remind you of Δf and Δx as the change in x and the corresponding change in f become smaller and smaller in the limit.

Before practicing how to use the definition of the derivative on examples, let us look at how it is related to the graph of f. First, note that the average rate of change of f

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

reminds us of the rise-over-run formula for the slope of a line. It is in fact the slope of a line: it is the slope of the line that passes through the points $P = (x_1, f(x_1))$ and $Q = (x_2, f(x_2))$:



Such a line is called a *secant line*. So the average rate of change of f between x_1 and x_2 is also the slope of the secant line to f between x_1 and x_2.

Here is an example of how you would find the equation of a secant line. Let $f(x) = \sqrt{x}$. Let us find the equation of the secant line between $x_1 = 4$ and $x_2 = 8$. The slope of the line is

$$\frac{\Delta f}{\Delta x} = \frac{\sqrt{4}-\sqrt{8}}{4-8} = \frac{\sqrt{8}-\sqrt{4}}{8-4} = \frac{\sqrt{8}-2}{4} = \frac{2\sqrt{2}-2}{4} = \frac{\sqrt{2}-1}{2}.$$

So the equation of the line is

$$y = mx + b = \frac{\sqrt{2}-1}{2}x + b$$

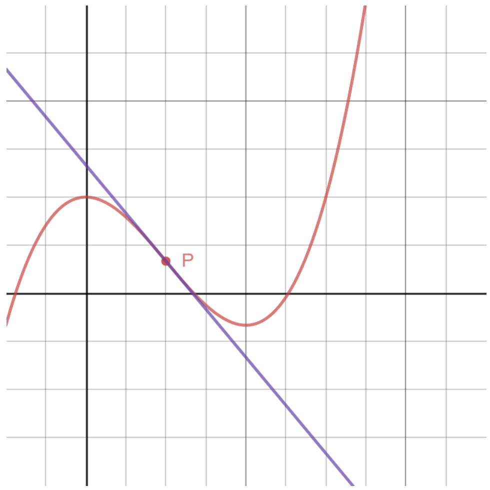
To find b, we note that the line passes through $(4, 2)$ (and also through $(8, \sqrt{8})$, but one point is enough). So

$$2 = \frac{\sqrt{2}-1}{2}4 + b = 2\sqrt{2} - 2 + b \implies b = 4 - 2\sqrt{2}.$$

Hence the equation of the secant line is

$$y = \frac{\sqrt{2}-1}{2}x + 4 - 2\sqrt{2}.$$

Now, try moving the point x_2 closer and closer to x_1 and visualize what happens to the secant line. You can actually try this with one of the Desmos apps posted on the class website. While you cannot just make $x_2 = x_1$, because then slope becomes $0/0$, which is undefined, you can make the distance between x_1 and x_2 really small. As you do, the secant line seems to no longer go through two points on the graph but only barely touch the graph in one point:



Such a line, one that just touches the graph of f at a point $(a, f(a))$ is called the *tangent line*. We got this line by moving the two points where a secant line crosses the graph of f closer and closer together. As we do, the slope of the secant line is the average rate of change of f over a shorter and shorter interval. It should come as no surprise that as $x_2 \to x_1$ and the secant line becomes the tangent line in the limit, the slope of that line becomes the limit of the difference quotient as $\Delta x \to 0$. That is the instantaneous rate of change of f or the derivative $f'(a)$. So $f'(a)$ is also the slope of the tangent line to f at a. In the examples below, you will see how you can find the equation of a tangent line. There are of course examples in the book too.

First, let $f(x) = 1/x$. We will find $f'(2)$ and the equation of the tangent line at $x = 2$. I will use the $h \to 0$ formula to find $f'(2)$. I could also use the $x \to 2$ formula.

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{2(2+h)} - \frac{2+h}{2(2+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{2(2+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{2h(2+h)} \\
&= \lim_{h\to 0} \frac{-1}{2(2+h)} && \text{we can cancel } h \text{ because } h \neq 0 \text{ as } h \to 0 \\
&= \lim_{h\to 0} \frac{-1}{4+2h} \\
&= -\frac{1}{4} && \text{by direct substitution into the rational function}
\end{aligned}$$

Now we know that the equation of the tangent line is

$$y = \frac{-1}{4}x + b.$$

To find b, we note that the line passes through $(2, 1/2)$ and so

$$\frac{1}{2} = -\frac{1}{4}2 + b = -\frac{1}{2} + b \implies b = 1.$$

Hence the equation of the tangent line is

$$y = -\frac{1}{4}x + 1.$$

Let $a \neq 0$. We will now find $f'(a)$ and the equation of the tangent line at $x = a$. I will use the $x \to a$ formula to find $f'(a)$. I could also use the $h \to 0$ formula.

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x\to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} \\
&= \lim_{x\to a} \frac{\frac{a}{xa} - \frac{x}{xa}}{x - a} \\
&= \lim_{x\to a} \frac{\frac{a-x}{xa}}{x - a} \\
&= \lim_{x\to a} \frac{\frac{-(x-a)}{xa}}{x - a} && \text{we can cancel } x - a \text{ because } x \neq a \text{ as } x \to a \\
&= \lim_{x\to a} \frac{-1}{xa} \\
&= \frac{-1}{aa} && \text{by direct substitution into the rational function} \\
&= -\frac{1}{a^2}
\end{aligned}$$

Now we know that the equation of the tangent line is

$$y = -\frac{1}{a^2}x + b.$$

To find b, we note that the line passes through $(a, 1/a)$ and so

$$\frac{1}{a} = -\frac{1}{a^2}a + b = -\frac{1}{a} + b \implies b = \frac{2}{a}.$$

Hence the equation of the tangent line is

$$y = -\frac{1}{a^2}x + \frac{2}{a}.$$

Let us try our new skills on the function $f(x) = \sqrt{x}$. First, let us find $f'(3)$ and the equation of the tangent line at $x = 3$. I will use the $x \to 3$ formula to find $f'(2)$. I could also use the $h \to 0$ formula.

$$\begin{aligned}
f'(3) &= \lim_{x\to a} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{x\to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}
\end{aligned}$$

We can now either factor the denominator as $x - 3 = (\sqrt{x} + \sqrt{3})(\sqrt{x} - \sqrt{3})$ or multiply both the numerator and the denominator by $\sqrt{x} + \sqrt{3}$. Either way, we want to note that $\sqrt{x} + \sqrt{3} \neq 0$, otherwise canceling it or multiplying both the numerator and the denominator by it would not be a legitimate step. The reason $\sqrt{x} + \sqrt{3} \neq 0$ is that $\sqrt{3}$ is a positive number and $\sqrt{x} \geq 0$, so

$\sqrt{x}+\sqrt{3} \geq \sqrt{3} > 0$. In fact, we know $x \to 3$, so $\sqrt{x}$ is a number that is close to $\sqrt{3}$, and so it is positive for sure. Therefore

$$\begin{aligned} f'(3) &= \lim_{x\to 3} \frac{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}{(x-3)(\sqrt{x}+\sqrt{3})} \\ &= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})} && \text{we can cancel } x-3 \text{ because } x \neq 3 \text{ as } x \to 3 \\ &= \lim_{x\to 3} \frac{1}{\sqrt{x}+\sqrt{3}} \end{aligned}$$

Now, we can either use the limit laws or argue that

$$\frac{1}{\sqrt{x}+\sqrt{3}}$$

is a continuous function at $x = 3$ and therefore we can substitute $x = 3$ to evaluate the limit. I will leave the argument with the limit laws to you. I will argue that $h(x) = \sqrt{x}$ is a root function that is continuous on its domain (which is $\mathbb{R}^{\geq 0}$), while $k(x) = \sqrt{3}$ and $l(x) = 1$ are constant functions and hence polynomials and also continuous on their domains (which is $\mathbb{R}$) by Theorem 6 in Section 1.5. So h, k, and l are all continuous on $\mathbb{R}^{\geq 0}$ and therefore

$$g(x) = \frac{l(x)}{h(x)+k(x)}$$

is also continuous at every $x \in \mathbb{R}^{\geq 0}$ that does not make the denominator 0 by the various parts of Theorem 4 in Section 1.5. Since $x = 3$ does not make $\sqrt{x}+\sqrt{3} = 0$, the function $g(x)$ is continuous at $x = 3$. Therefore we can use direct substitution to find

$$f'(3) = \lim_{x\to 3} \frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

The equation of the tangent line is then

$$y = \frac{1}{2\sqrt{3}}x + b.$$

To find b, we note that the line passes through $(3, \sqrt{3})$ and so

$$\sqrt{3} = \frac{1}{2\sqrt{3}}3 + b = \frac{\sqrt{3}}{2} + b \implies b = \sqrt{3} - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

Hence the equation of the tangent line is

$$y = \frac{1}{2\sqrt{3}}x + \frac{\sqrt{3}}{2}.$$

Let $a > 0$. We will now find $f'(a)$ and the equation of the tangent line at $x = a$. I will use the $h \to 0$ formula to find $f'(a)$. I could also use the $x \to a$ formula.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \end{aligned}$$

We will multiply the numerator and the denominator by $\sqrt{a+h}+\sqrt{a}$. We can only do that if we know $\sqrt{a+h}+\sqrt{a} \neq 0$. We do know $a > 0$, and so $\sqrt{a} > 0$. As $h \to 0$, $a+h$ is getting close to a, and hence $\sqrt{a+h}$ is getting close to $\sqrt{a}$. The important thing is that if x is close enough to 0, then $a+h \geq 0$, and so we do not have a negative number under the square root. Hence $\sqrt{a+h} \geq 0$ and $\sqrt{a+h}+\sqrt{a} \geq \sqrt{a} > 0$. So we can say

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})} \\ &= \lim_{h\to 0} \frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{a+h}+\sqrt{a})} && \text{we can cancel } h \text{ because } h \neq 0 \text{ as } h \to 0 \\ &= \lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} \end{aligned}$$

We can now argue that

$$\lim_{h\to 0} \sqrt{a+h} = \sqrt{\lim_{h\to 0}(a+h)} = \sqrt{a}$$

by LL11 and direct substitution into the polynomial;

$$\lim_{h\to 0} \sqrt{a} = \sqrt{a} \quad \text{and} \quad \lim_{h\to 0} 1 = 1$$

by LL7;

$$\lim_{h\to 0}(\sqrt{a+h}+\sqrt{a}) = \lim_{h\to 0} \sqrt{a+h} + \lim_{h\to 0} \sqrt{a} = 2\sqrt{a}$$

by LL1; and finally,

$$\lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{\lim_{h\to 0} 1}{\lim_{h\to 0}(\sqrt{a+h}+\sqrt{a})} = \frac{1}{2\sqrt{a}}$$

by LL5. The equation of the tangent line is then

$$y = \frac{1}{2\sqrt{a}}x + b.$$

To find b, we note that the line passes through $(a, \sqrt{a})$ and so

$$\sqrt{a} = \frac{1}{2\sqrt{a}}a + b = \frac{\sqrt{a}}{2} + b \implies b = \sqrt{a} - \frac{\sqrt{a}}{2} = \frac{\sqrt{a}}{2}.$$

Hence the equation of the tangent line is

$$y = \frac{1}{2\sqrt{a}}x + \frac{\sqrt{a}}{2}.$$

If you graph the functions and their tangent lines above in Desmos, you will see that steepness of the line is a close fit to the steepness of the graph at the point of tangency. For this reason, the derivative $f'(a)$ is also referred to as the *slope of f at a*. There is an important point here. The tangent line to f at a is a linear function whose rate of change at a is exactly the same as the instantaneous rate of change of f. So the tangent line behaves the way f would if f were a linear function. Of course, f is typically not a linear function and does not really behave like one. As you move away from the point a, the graph of f and the tangent line go on their separate ways. But if you look close to a, they look similar. The closer you look, the closer the line and the graph appear. Try this in Desmos: zoom in on the point a. So the tangent line allows us to approximate the behavior of the function f with a linear function near the point a. While the tangent line has the same slope everywhere on the line, the graph of a typical function f does not. This is why the approximation is only local: it only works well near the point a.

Example 6 in the text illustrates what the derivative means in real life on the national debt function $D(t)$. If $D(t)$ is the federal debt (measured in billions of dollars) in the year t, then $D'(2000)$ was the rate of change of the federal debt in 2000. This means that in 2000, the federal debt would have increased by *approximately* $D'(t)$ dollars. The reason this is approximate is that D is not a linear function, so its change over a period of one year is not going to be exactly the same as its instantaneous rate of change at exactly the point $t = 2000$, which is presumably the moment the year 2000 began. The example in fact does use the average rate of change of D between 1995 and 2005 to estimate the instantaneous rate of change at $t = 2000$:

$$D'(2000) \approx \frac{D(2005) - D(1995)}{2005 - 1995} = \frac{7932.7 - 4974.0}{10} = 295.87\,\text{billion}\,\frac{\$}{\text{year}},$$

except it computes this in a less direct way by averaging the average rates of change between 1995 and 2000 and between 2000 and 2005. If you are curious why that is the same thing, here is that calculation:

$$\begin{aligned}
D'(2000) &\approx \frac{\frac{D(2005)-D(2000)}{2005-2000} + \frac{D(2000)-D(1995)}{2000-1995}}{2} \\
&= \frac{1}{2}\left(\frac{D(2005) - D(2000)}{5} + \frac{D(2000) - D(1995)}{5}\right) \\
&= \frac{1}{2}\,\frac{D(2005) - D(2000) + D(2000) - D(1995)}{5} \\
&= \frac{1}{2}\,\frac{D(2005) - D(1995)}{5} \\
&= \frac{D(2005) - D(1995)}{10} \\
&= \frac{D(2005) - D(1995)}{2005 - 1995}.
\end{aligned}$$