Notes for Section 2.2

We have seen the definition of the derivative of a function f at a point a and what the derivative means in terms of the slope of the tangent line to f at a and the steepness of the graph at $x = a$. In this section, we will treat a as a variable and $f'(a)$ as a function of a. Of course, we could denote the input to f by any letter we want, not only a. We often use x as the input to a function whose domain and codomain are sets of real numbers. The only thing to be careful about is that we calculate f' using the $x \to a$ formula, we cannot replace a with x, otherwise we would get

$$f'(x) = \lim_{x \to x} \frac{f(x) - f(x)}{x - x},$$

which makes no sense. But once we have evaluated

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

f' should be all in terms of a–there definitely should not be any x left in it–and we can now substitute x for a if we wish. E.g. if $f(x) = x^2$ then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x-a)(x+a)}{x - a} = \lim_{x \to a}(x + a) = 2a.$$

Now we can say $f'(x) = 2x$. One reason we may want to do this is to be able to plot f and f' in the same xy-coordinate system.

The same issue does not arise when using the $h \to 0$ formula for the derivative. It is perfectly meaningful to calculate

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Examples 1-4 in Section 2.2 illustrate things we already did in Section 2.1. You can use them to review how to estimate the derivative from the graph of f and how to find the derivative of f by using the limit of the difference quotient. I showed you in class how you can use the graph of a function to sketch the graph of the derivative by sliding the tangent line along the graph and plotting how the slope of the tangent line changes along the way. This is hard explain in writing without actually doing it on the whiteboard. You can practice this skill by using one of the Desmos tangent line apps linked from the class website. I also showed you how to do the reverse: sketch the graph of f given the graph of f'. There is no app that I know of that demonstrates this skill, but if you understand how to sketch the graph of f' based on the graph of f, you should be able to do the reverse too. So focus on that skill first.

We have seen the notations

$$f'(a),\ \left.\frac{df}{dx}\right|_{x=a},\ \left.\frac{df}{dx}\right]_{x=a}$$

for the derivative of f at a. If we want to refer to the entire derivative function rather than its value at one specific point, we can use any of

$$f' = \frac{df}{dx} = \frac{d}{dx}f = \frac{d}{dx}f(x) = \frac{dy}{dx} = D\,f = D\,f(x) = D_x f = D_x f(x).$$

They all mean the same thing. The $\frac{d}{dx}$, D, and D_x mean "differentiate what follows." They are called operators. An operator is a function that takes a function as its input and returns another function (in this case the derivative of the input) as its output. The $\frac{dy}{dx}$ notation is motivated by the fact that when we graph the function f we plot of the values of f on the y-axis. We will use the f' and the $\frac{df}{dx}$ notations, and occasionally the $\frac{dy}{dx}$ notation. Our textbook does not use $D\,f$ and $D_x f$ notations, but some other textbooks do.

Do not confuse the derivative function f' with the derivative $f'(a)$ at a specific point $x = a$. The former is a function, while the latter is a number. Both are called the derivative of f, and the context (and the notation) tells you which one we are referring to.

If you want to refer to the derivative of f at a point a, you can use

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right]_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right]_{x=a} = [D\,f](a) = [D_x f](a).$$

Avoid

$$\frac{d}{dx} f(a)$$

because it means the derivative of $f(a)$ with respect to changes in the variable x. But $f(a)$ is a constant number, whose value depends on a but not on x, and therefore its rate of change with respect to changes in x is 0. It is for this same reason that we have brackets in $[D\,f](a)$ and $[D_x f](a)$.

We have been talking about finding the derivative of a function f. Another term for doing that is to *differentiate* f. So we differentiate f to find f', and we differentiate f at a to find $f'(a)$. Do not say derive f. That is incorrect terminology.

The derivative of a function f at a need not exist. It is a limit and we have certainly seen limits that do not exist. When $f'(a)$ does exist, we say f *is differentiable at* a, and when $f'(a)$ does not exist, we say f *is nondifferentiable at* a. Here is the formal definition.

Definition. Let f be a function of real numbers. For a real number a, we say f *is differentiable at* a if $f'(a)$ exists. If $f'(a)$ does not exist, we say f *is nondifferentiable at* a. For an open interval (a, b), f is called *differentiable on* (a, b) if f is differentiable at every $x \in (a, b)$.

The interval in the definition above is an open interval because if it had a sharp endpoint, like a closed interval $[a, b]$ or a half-open, half-closed interval $[a, b)$ has at a, then we could not look at the two-sided limit of the difference quotient there.

Here are a few typical ways that a function can fail to be differentiable. These are certainly not all of the ways a function can be nondifferentiable at a point, but studying these will give you a good idea of why the limit of the difference quotient may fail to exist. Graph the functions in Desmos to be able to better follow the explanation.

1. The function $f(x) = \frac{x^2-4}{x-2}$ has a hole (a kind of removable discontinuity) in the graph at $x = 2$ because the denominator is 0 there. Remember that Desmos does not indicate such discontinuities, but if you click there with the mouse, it will show you that the value of f is undefined at $x = 2$. Since $f(2)$ does not exist, the difference quotient

$$\frac{f(x) - f(2)}{x - 2}$$

does not exist either. Hence

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

cannot exist either.

2. The function $f(x) = 1/x$ has an infinite discontinuity at $x = 0$ because the denominator is 0 there. Since $f(0)$ does not exist, the difference quotient

$$\frac{f(x) - f(0)}{x - 0}$$

does not exist either. Hence

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

cannot exist either.

3. The piecewise defined function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

has a jump discontinuity at $x = 0$. You can graph piecewise defined functions in Desmos using the right syntax. Google "piecewise function desmos." You would enter this one as $\mathtt{f(x)} = \{\mathtt{x} \geq \mathtt{0} : \mathtt{1},\ \mathtt{x} < \mathtt{0} : -\mathtt{1}\}$. The derivative at 0 is

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 1}{x}.$$

If $x > 0$, then $f(x) = 1$, so the right limit is

$$\lim_{x \to 0^+} \frac{f(x) - 1}{x} = \lim_{x \to 0^+} \frac{1 - 1}{x} = \lim_{x \to 0^+} 0 = 0.$$

If $x < 0$, then $f(x) = -1$, and so the left limit is

$$\lim_{x \to 0^-} \frac{f(x) - 1}{x} = \lim_{x \to 0^-} \frac{-1 - 1}{x} = \lim_{x \to 0^-} -\frac{2}{x} = \infty$$

since x a negative number, which makes the value of $-2/x$ positive, and this value gets larger and larger as x gets closer and closer to 0. So the two-sided limit does not exist either.

It is worth looking at this example in one of the Desmos apps that will draw secant lines (there is one linked from the class website). Look at the secant line between $x_1 = 0$ and $x_2 \neq 0$. If $x_2 > 0$ then the secant line is always horizontal. But if you move x_2 to the negative side of the x-axis, then the secant line has a positive slope and the slope gets steeper and steeper as you move x_2 closer to 0. So the secant line is closer and closer to a vertical line as $x_2 \to 0^-$. In any case, there is no tangent line that the secant lines (on the positive and negative sides of the x-axis) get close to as $x_2 \to 0$.

4. The piecewise defined function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has a removable discontinuity at $x = 0$. You should be able to graph this function by hand. (Graph it in Desmos as $\mathtt{f(x)} = \{\mathtt{x} < \mathtt{0} : \mathtt{1},\ \mathtt{x} > \mathtt{0} : \mathtt{1},\ \mathtt{x} = \mathtt{0} : \mathtt{0}\}$. The derivative at 0 is

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 0}{x} = \lim_{x \to 0} \frac{f(x) - 0}{x}.$$

Since $x \neq 0$ as $x \to 0$, $f(x) = 1$ in the difference quotient. So

$$f'(0) = \lim_{x \to 0} \frac{1 - 0}{x} = \lim_{x \to 0} \frac{1}{x}.$$

This limit clearly does not exist as $1/x$ either gets large (and positive) or very negative as x gets close to 0. The left limit is $-\infty$ and the right limit is ∞.

This is also worth looking at in a Desmos app that will draw secant lines. Look at the secant line between $x_1 = 0$ and $x_2 \neq 0$. If $x_2 > 0$ then the secant line has a positive slope. As you move x_2 closer and closer to 0, the slope of the secant line becomes steeper and steeper and the secant line gets closer and closer to looking like a vertical line. If x_2 is on the negative side of the x-axis, then the secant line has a negative slope. That slope also gets steeper and steeper as you move x_2 closer to 0, and the secant line is closer and closer to a vertical line. In any case, the slope of secant line either gets really large (if $x_2 > 0$) or really negative (if $x_2 < 0$) and does not get close a number.

5. So far all our examples were discontinuous at a. But a function does not have to be discontinuous to be nondifferentiable. An example of a function which is continuous but still nondifferentiable is $f(x) = |x|$ at 0.
$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|}{x}.$$
We have seen this before as an example of a limit that does not exist because the left and the right limits are different:
$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$
$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$
Graph this in a Desmos app that will draw secant lines. Looking at the secant line between $x_1 = 0$ and $x_2 \neq 0$, you will see that the secant line is always $y = x$ if $x_2 > 0$ and $y = -x$ if $x_2 < 0$. So there is no one tangent line that the secant lines get close to as $x_2 \to 0$.
Here is another way to think about this. For a function that does have a tangent line at a point a, the tangent line approximates the behavior of the function near a. For example, try this with the function $f(x) = x^2$ at any number a. Graph the function in a Desmos app that will draw tangent lines. Zoom in more and more and more about the point $(a, f(a))$. You will see that the graph of f and the tangent line look more and more alike. Now, graph $f(x) = |x|$ and try zooming in on $(0, 0)$. No matter how much you zoom in, the graph never looks like a piece of a straight line. The break in the graph at $(0, 0)$ always remains a sharp vertex. So no line can approximate the behavior of $f(x) = |x|$ at $x = 0$. Such a sharp vertex is called a *cusp*. A function whose graph has a cusp is never differentiable at that cusp for much the same reason that we described for the absolute value function at $x = 0$.
6. The function $f(x) = \sqrt[3]{x}$ is nondifferentiable at 0.
$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}} = \lim_{x\to 0} \frac{1}{\sqrt[3]{x^2}}.$$
If x is a number close to 0, then x^2 is also a number close to 0. Note that x^2 is positive, whether x is positive or negative. So $\sqrt[3]{x^2}$ is also a small (close to 0) positive number. Therefore $1/\sqrt[3]{x^2}$ is a large positive number and gets larger and larger as x gets closer and closer to 0. Therefore
$$\lim_{x\to 0} \frac{1}{\sqrt[3]{x^2}} = \infty,$$
and $f'(0)$ does not exist.
Graph $f(x) = \sqrt[3]{x}$ in a Desmos app that will draw secant lines. Looking at the secant line between $x_1 = 0$ and $x_2 \neq 0$, you will see that the secant line has a positive slope. As you move x_2 closer and closer to 0, the slope of the secant line becomes steeper and steeper and the secant line gets closer and closer to looking like a vertical line. This is so whether x_2 is positive or negative. We can say that the vertical line $x = 0$, which the secant lines get closer and closer to is the tanget line to $f(x) = \sqrt[3]{x}$ at $x = 0$. So the function does have a tangent line at $x = 0$, but the tangent line is vertical and therefore its slope is undefined.

A number of the examples of nondifferentiable functions we have just seen were discontinuous. We may suspect that a function f that is discontinuous at a point a must also be nondifferentiable at a. This is indeed so and we are about to give a rigorous argument for it. But note that the converse is certainly not true: if the function f is nondifferentiable at a it need not be discontinuous there. This was illustrated by the last two examples, the one with the cusp and the one with the vertical tangent line.

Before giving a formal proof, here is the intuition in this. In

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

the denominator $x - a$ becomes a number close to 0 as $x \to a$. For the value of the difference quotient to have a limit, the ratio

$$\frac{f(x) - f(a)}{x - a}$$

has to get closer and closer to some number L as x gets close to a. Therefore

$$\frac{f(x) - f(a)}{x - a} \approx L \implies f(x) - f(a) \approx L(x - a).$$

Since $x - a$ is getting closer and closer to 0, $f(x) - f(a)$ also needs to get closer and closer to 0 as $x \to a$. So $f(x)$ must be getting closer and closer to $f(a)$. That is exactly what it means for f to be continuous at a. The actual proof is just a somewhat more rigorous version of this intuitive argument.

Theorem. *Let f be a function of real numbers and let $a \in \mathbb{R}$. If f is differentiable at a then f is also continuous at a.*

Proof: Suppose f is differentiable at a. Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. We want to show that f is continuous at a, that is $\lim_{x \to a} f(x) = f(a)$.

We will first prove that

$$\lim_{x \to a} [f(x) - f(a)] = 0.$$

Since $x \neq a$ as $x \to a$, we know that $x - a \neq 0$. Therefore we can multiply and divide $f(x) - f(a)$ by the nonzero number $x - a$ without changing its value:

$$\begin{aligned}
\lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} (x - a) \right] \\
&= \underbrace{\left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right]}_{= f'(a)} \left[\lim_{x \to a} (x - a) \right] && \text{by LL4} \\
&= f'(a)(a - a) && \text{by direct subsitution into the polynomial } x - a \\
&= 0
\end{aligned}$$

Now, we will show that $\lim_{x \to a} f(x) = f(a)$. Notice that if we subtract $f(a)$ from $f(x)$ and then add $f(a)$, that does not change the value ofthe function inside the limit:

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} [f(x) - f(a) + f(a)] \\
&= \underbrace{\lim_{x \to a} [f(x) - f(a)]}_{=0} + \lim_{x \to a} f(a) && \text{by LL1} \\
&= 0 + f(a) && \text{by LL7} \\
&= f(a)
\end{aligned}$$

This is exactly what we wanted to show. $\square$

This is an important theoretical result and underscores the significance of continuity in calculus.

The last topic in this section is higher derivatives. The derivative of a function f is another function f'. We can of course look at the derivative of f' too. The derivative of f' is yet another function $(f')'$. It is usually denoted by f'' to simplify the notation. It is called the second deruvative of f for obvious reasons. There is no reason to stop at the second derivative, we can keep differentiating to get the third, fourth, etc derivative of f. If we differentiate n times, we get the n-th derivative of f. Here are the various notations used for these:

$$\begin{aligned} f'' &= \frac{d^2f}{dx^2} = \frac{d^2y}{dx^2} = D^2f = D_x^2f \\ f''' &= \frac{d^3f}{dx^3} = \frac{d^3y}{dx^3} = D^3f = D_x^3f \\ f'''' &= \frac{d^4f}{dx^4} = \frac{d^4y}{dx^4} = D^4f = D_x^4f \\ f^{(n)} &= \frac{d^nf}{dx^n} = \frac{d^ny}{dx^n} = D^nf = D_x^nf \end{aligned}$$

Notice that in the n-th derivative, the n in the superscript on f is in parentheses, to distinguish the n-th derivative of f from f^n, which means either the n-th power of $f^n(x) = [f(x)]^n$ or f composed with itself n times, $f^n = f \circ f \circ \cdots \circ f$ depending on the context. The prime notation $'$ is practical only up to about the 4th derivative. There is no set rule, and it is perfectly fine to use either f'''' or $f^{(4)}$, but you would not want to use the prime notation to denote the 100th derivative of f.

Also, note that in $\frac{d^2f}{dx^2}$ the exponent 2 is on the d in the numerator but on the x in the denominator. There is some logic to this. To calculate $\frac{d^2f}{dx^2}$, you would differentiate the first derivative $\frac{df}{dx}$. That is

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right).$$

Notice that there are formally two d's in the numerator but only one f. This is why the numerator is d^2f. But there actually are two x's in the denominator. Well, there are also two d's down there, so the denominator should be $(dx)^2$. That in fact is what dx^2 stands for, we just do not write the parentheses to keep the notation simpler. The notation for the higher derivatives follows the same logic.

What these higher derivatives mean in practical applications depends on the context. Recall the example of the ball dropped from the observation deck of the CN Tower from the previous section. The position of the ball in terms of the distance it has fallen from its initial position was given by $f(t) = 4.9t^2$ where f is measured in meters and t is in seconds. We calculated that the ball's velocity at time t is $f'(t) = 9.8t$ and noted that its units are m/s. The second derivative of f is

$$\begin{aligned} f''(t) &= \lim_{h\to 0} \frac{f'(t+h) - f'(t)}{h} \\ &= \lim_{h\to 0} \frac{9.8(t+h) - 9.8t}{h} \\ &= \lim_{h\to 0} \frac{9.8t + 9.8h - 9.8t}{h} \\ &= \lim_{h\to 0} \frac{9.8h}{h} \\ &= \lim_{h\to 0} 9.8 \\ &= 9.8 \end{aligned}$$

by LL7. This tells us the rate at which the velocity of the falling ball is increasing. That is how fast the ball is accelerating. The units confirm this: since the units of f' are m/s and the units of

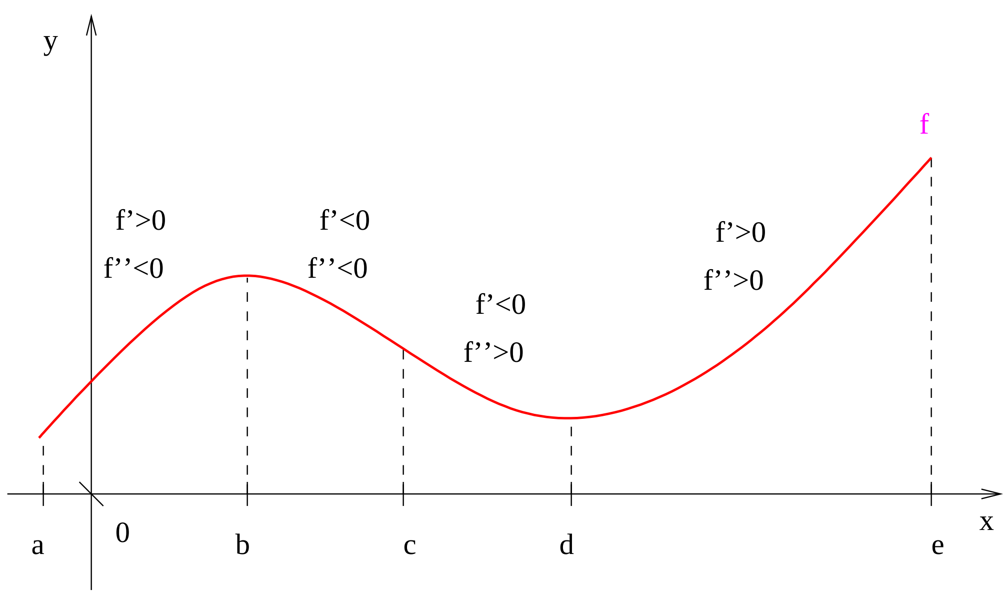
t are seconds, the units of f'' are (m/s)/s = m/s^2. In fact, you have probably already recognized that $9.8\frac{\text{m}}{\text{s}^2}$ is the standard gravitational acceleration on the surface of earth. In general, if $f(t)$ is the position of a moving object at time t, f' is the object's velocity and f'' is its acceleration. This is also a good place to explain the difference between velocity and speed. Speed is the magnitude of velocity. For us in single variable calculus, that means speed is the absolute value of velocity. The velocity of an object can be positive (if moving forward) or negative (if moving backwards). But speed is never negative. For motion in two or more dimensions, velocity has a direction. Speed does not have direction. It is just a number that measures how quickly an object moves in whatever direction it is moving. The speedometer in a car really measures speed. It tells you how fast the car is moving but does not care about what direction the car is moving.

In general, the second derivative of a function f tells us how quickly the rate of change of f changes. That is some kind of acceleration, although the exact meaning depends on the context. For example, if $p(t)$ is the global population in year t, then $p'(t)$ is the rate of population growth, and if the rate of population growth is increasing, then we would say that population growth is accelerating and $p''(t)$ would measure how quickly it accelerates.

The practical meaning of higher derivatives is often murkier. If $f(t)$ is the position of a moving object at time t then $f'''(t)$ is the rate of change of acceleration. Just for curiosity, that is called jerk. Think about going bungee jumping with a regular rope instead of a rubber cord.

From the point of view of graphs, we have seen in class that the sign of f' tells us whether f is increasing or decreasing. Where the graph of f is climbing, f' is positive, and where the graph of f is descending, f' is negative. We have also seen that the sign of f'' is linked to which way the graph of f is curving. Wherever the graph of f is concave up (curving up), f'' is positive. Note that f could be concave up whether f is increasing or decreasing. Wherever the graph of f is concave down (curving down), f'' is negative. The reason is quite intuitive. If $f'' > 0$ then f' is increasing. That means either that the graph of f is climbing (if $f' > 0$) and is doing so more and more steeply as we go from left to right, or that the graph of f is descending (if $f' < 0$) and is doing so less and less steeply as we go from left to right. If $f'' < 0$ the situation is reversed. Now f' is decreasing. That means either that the graph of f is descending (if $f' < 0$) and is doing so more and more steeply as we go from left to right, or that the graph of f is climbing (if $f' > 0$) and is doing so less and less steeply as we go from left to right.

Here is a graph to help you remember this:



We will circle back to these ideas in Chapter 3.