Notes for Section 2.3

We have learned the meaning of the derivative of a function and how to calculate it the hard way, by finding the limit of the difference quotient. We can now develop shortcuts for finding derivatives quickly and easily, much the same way we developed a variety of shortcuts to find limits quickly and easily, such as the limit laws, the direct substitution property, and the Squeeze Theorem. That is what this section and the next few after this one are about.

We will start with the constant function $f(x) = c$. Even without doing any calculations, we can easily tell that the derivative of this function is 0 at any point a. This is because the value of this function does not change as x changes, so its rate of change should be 0. Another intuitive way to see this is to consider the graph of f. It is a horizontal line. The tangent line to this graph at any point a is the same horizontal line. Its slope is 0. Here is the actual calculation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

by LL7. We can call this the Constant Function Rule and state it as a theorem.

Theorem (The Constant Function Rule)**.** *For any $c \in \mathbb{R}$,*

$$\frac{d}{dx} c = 0.$$

The next kind of functions we will consider are power functions. Let $f(x) = x^n$ where n is a positive integer. Then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}.$$

We have worked this out in the special case when $n = 2$ by factoring the numerator as $x^2 - a^2 = (x - a)(x + a)$ and then canceling the $x - a$ factor from the numerator and the denominator. The same idea works in general. How to factor $x^n - a^n$ is less obvious, but it is not difficult to see the pattern if we look at a few such factorings. You can verify each one by distributing them out. In fact, if your memory of these from algebra class is rusty, you should distribute them to understand why they work.

$$\begin{aligned}
x^2 - a^2 &= (x - a)(x + a) \\
x^3 - a^3 &= (x - a)(x^2 + xa + a^2) \\
x^4 - a^4 &= (x - a)(x^3 + x^2a + xa^2 + a^3) \\
x^5 - a^5 &= (x - a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)
\end{aligned}$$

That should be enough to see the pattern. We can easily verify that for every $n \in \mathbb{Z}^+$,

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$$

by distributing first the x and then the $-a$:

$$\begin{aligned}
&(x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) \\
&= x^n + x^{n-1}a + x^{n-2}a^2 + \cdots + x^2a^{n-2} + xa^{n-1} \\
&\qquad - x^{n-1}a - x^{n-2}a^2 + \cdots - x^2a^{n-2} - xa^{n-1} - a^n \\
&= x^n - a^n
\end{aligned}$$

You could have gotten the same result by using long-hand polynomial division to divide $x^n - a^n$ by $x - a$, if you remember how to do that from algebra.

We are now ready to evaluate the limit:

$$\begin{aligned}
\lim_{x\to a}\frac{x^n-a^n}{x-a} &= \lim_{x\to a}\frac{(x-a)(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1})}{x-a} \\
&= \lim_{x\to a}(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1}) \\
&= a^{n-1}+a^{n-2}a+a^{n-3}a^2+\cdots+aa^{n-2}+a^{n-1} \\
&= \underbrace{a^{n-1}+a^{n-1}+a^{n-1}\cdots+a^{n-1}+a^{n-1}}_{n\text{ terms}} \\
&= na^{n-1},
\end{aligned}$$

where we could cancel $x-a$ because it is not 0 as $x\to a$ and we could use direct substitution to evaluate the last limit because $x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1}$ is a polynomial in x. Hence the derivative is $f'(a)=na^{n-1}$, or with x as the variable, $f'(x)=nx^{n-1}$. We can refer to this as the Power Function Rule and state this as a theorem.

Theorem (The Power Function Rule)**.** *For all $n\in\mathbb{Z}^+$,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

The book takes a slightly different approach to derive it by using the $h\to 0$ formula for the derivative and the Binomial Theorem. This works just as well if you are familiar with the Binomial Theorem, but I am not sure you all are.

We showed that the Power Function Rule holds for every positive integer exponent n. Actually, it holds for any real number exponent, and we will be able to show this later. For now, feel free to use it.

Theorem (The Power Function Rule, general version)**.** *For all $n\in\mathbb{R}$,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Here are a few examples

$$\begin{aligned}
\frac{d}{dx}x^{17} &= 17x^{17-1} = 17x^{16} \\
\frac{d}{dx}\frac{1}{x} &= x^{-1} = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2} \\
\frac{d}{dx}\frac{1}{x^5} &= x^{-5} = -5x^{-5-1} = -5x^{-6} = -\frac{5}{x^6} \\
\frac{d}{dx}\sqrt{x} &= x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \\
\frac{d}{dx}\sqrt[3]{x} &= x^{1/3} = \frac{1}{3}x^{1/3-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}} \\
\frac{d}{dx}\frac{1}{\sqrt{x}} &= x^{-1/2} = -\frac{1}{2}x^{-1/2-1} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}} = -\frac{1}{2\sqrt{x}^3} \\
\frac{d}{dx}x &= \frac{d}{dx}x^1 = 1\cdot x^{1-1} = x^0 = 1
\end{aligned}$$

The last example is a little subtle. In fact, $x^0=1$ whenever $x\neq 0$. But 0^0 is undefined and is not equal to 1. Yet, if you think about the graph of the function $f(x)=x$, its graph is a line of slope 1. The tangent line to this graph at any point, including $x=0$ is the same line $y=x$. So the derivative of $f(x)=x$ should be 1 everywhere. Nothing unusual happens at $x=0$. In fact, you could easily use the definition of the derivative to conlude the same (try it!). So the correct interpretation of

$$\frac{d}{dx}x = x^0$$

is in fact that

$$\frac{d}{dx}x = 1.$$

As a curiosity, the Power Function Rule even works for x^0:

$$\frac{d}{dx}x^0 = 0x^{0-1} = 0\,\frac{1}{x} = 0 \quad \text{unless } x = 0.$$

Remember that $x^0 = 1$ for all real numbers $x \neq 0$. So x^0 is a constant function on the domain $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, it makes sense that its derivative is 0. The hole at $x = 0$ is a bit of a red herring. It is a removable discontinuity: the function $g(x) = 1$ on the domain $\mathbb{R}$ is continuous and $g(x) = f(x)$ for all $x \neq 0$. We already know that $g'(x) = 0$ for all $x \in \mathbb{R}$.

Do not confuse power functions with exponential functions. For a power function, the variable is in the base and the exponent is constant. For an exponential function, the variable is in the exponent and the base is constant. The Power Function Rule does not work expontential functions:

$$\frac{d}{dx}2^x \neq x2^{x-1}.$$

There is of course an appropriate rule for the derivatives of exponential functions, but we do not know it yet. For now, the important thing is that you do not misuse the Power Function Rule for an exponential function.

Now that we know how to differentiate power functions, we would like to do the same for polynomials, such as $f(x) = 5x^3 - 3x^2 + 10$. We can do this by learning how to deal with sums, differences, and constant multiples of functions in general. These rules are quite straightforward.

Theorem (The Sum Rule)**.** *Let f and g be functions of real numbers and $a \in \mathbb{R}$. If f and g are both differentiable at a, then $f + g$ is also differentiable at a and*

$$(f + g)'(a) = f'(a) + g'(a).$$

Proof: Suppose f and g are both differentiable at a.

$$\begin{aligned}
(f+g)'(a) &= \lim_{x\to a} \frac{(f+g)(x) - (f+g)(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x) + g(x) - f(a) - g(a)}{x-a} \\
&= \lim_{x\to a} \left[\frac{f(x) - f(a)}{x-a} + \frac{g(x) - g(a)}{x-a}\right] \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} + \lim_{x\to a} \frac{g(x) - g(a)}{x-a} && \text{by LL1} \\
&= f'(a) + g'(a)
\end{aligned}$$

Since f and g are differentiable at a, $f'(a)$ and $g'(a)$ both exist. Therefore $(f + g)'(a) = f'(a) + g'(a)$ also exists, which makes $f + g$ differentiable at a. This is exactly what we wanted to show. □

You can also express the Sum Rule as

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

or

$$(f+g)' = f' + g'.$$

These both say that the derivative function of $f+g$ is the sum of the derivative functions of f and g at whatever inputs that makes sense.

Theorem (The Difference Rule)**.** *Let f and g be functions of real numbers and $a \in \mathbb{R}$. If f and g are both differentiable at a, then $f-g$ is also differentiable at a and*

$$(f-g)'(a) = f'(a) - g'(a).$$

This follows from Limit Law 2 much the same way as the Sum Rule follows from Limit Law 1. I will leave the details to you. Try writing down the argument so you practice how to do it.

You can also express the Difference Rule as

$$\frac{d}{dx}(f-g) = \frac{df}{dx} - \frac{dg}{dx}$$

or

$$(f-g)' = f' - g'.$$

Theorem (The Constant Multiple Rule)**.** *Let f be a function of real numbers and $a \in \mathbb{R}$. If f is differentiable at a, then for any constant $c \in \mathbb{R}$, cf is also differentiable at a and*

$$(cf)'(a) = f'(a) - g'(a).$$

Proof: Suppose f is differentiable at a.

$$\begin{aligned}
(cf)'(a) &= \lim_{x\to a} \frac{(cf)(x) - (cf)(a)}{x-a} \\
&= \lim_{x\to a} \frac{cf(x) - cf(a)}{x-a} \\
&= \lim_{x\to a} \left[c\,\frac{f(x)-f(a)}{x-a} \right] \\
&= c \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{by LL3} \\
&= cf'(a)
\end{aligned}$$

Since f is differentiable at a, $f'(a)$ exists. Therefore $(cf)'(a) = cf'(a)$ also exists, which makes cf differentiable at a. This is exactly what we wanted to show. □

You can also express the Constant Multiple Rule as

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

or

$$(cf)' = cf'.$$

In fact, now that we proved the Sum and the Constant Multiple Rules, we could easily prove the Difference Rule by

$$\begin{aligned}
(f-g)'(a) &= f'(a) + (-g)'(a) && \text{by the Sum Rule} \\
&= f'(a) + \big((-1)g\big)'(a) \\
&= f'(a) + (-1)g'(a) && \text{by the Constant Multiple Rule} \\
&= f'(a) - g'(a)
\end{aligned}$$

Here is an example how you can use these rules to easily find the derivative of the function $f(x) = 7x^4 - 2\sqrt[4]{x} + 5$:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(7x^4 - 2\sqrt[4]{x} + 5) \\
&= \frac{d}{dx}(7x^4) - \frac{d}{dx}(2\sqrt[4]{x}) + \frac{d}{dx}5 && \text{by the Sum and Difference Rules} \\
&= 7\frac{d}{dx}x^4 - 2\frac{d}{dx}x^{1/4} + \frac{d}{dx}5 && \text{by the Constant Multiple Rule} \\
&= 7(4x^3) - 2\,\frac{1}{4}x^{1/4-1} + \frac{d}{dx}5 && \text{by the Power Function Rule} \\
&= 28x^3 - \frac{1}{2}x^{-3/4} + 0 && \text{by the Constant Function Rule} \\
&= 28x^3 - \frac{1}{2\sqrt[4]{x^3}}
\end{aligned}
$$

Two other functions whose derivatives we can figure out with our current tools are the sine and the cosine functions.

First, recall that we learned in Section 1.4 that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Notice that this is actually the derivative of the sine function at 0. If $f(x) = \sin(x)$, then

$$f'(0) = \lim_{x\to 0} \frac{\sin(x) - \sin(0)}{x - 0} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

We also learned back in Section 1.4 that

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0.$$

This is actually the derivative of the cosine function at 0. If $f(x) = \cos(x)$, then

$$f'(0) = \lim_{x\to 0} \frac{\cos(x) - \cos(0)}{x - 0} = \lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0.$$

To find the derivative of $f(x) = \sin(x)$ in general, we can use the definition of the derivative with $h \to 0$:

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

We use the addition formula for the sine function:

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h).$$

So

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \left[\sin(x)\,\frac{\cos(h)-1}{h} + \cos(x)\,\frac{\sin(h)}{h}\right] \\
&= \lim_{h\to 0} \left[\sin(x)\,\frac{\cos(h)-1}{h}\right] + \lim_{h\to 0}\left[\cos(x)\,\frac{\sin(h)}{h}\right] && \text{by LL1} \\
&= \sin(x)\,\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\,\lim_{h\to 0}\frac{\sin(h)}{h} && \text{by LL3} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x)
\end{aligned}$$

We can find the derivative of $f(x) = \cos(x)$ similarly:

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}.$$

We now use the addition formula for the cosine function:

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h).$$

So

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x) - \sin(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \left[\cos(x)\,\frac{\cos(h)-1}{h} - \sin(x)\,\frac{\sin(h)}{h}\right] \\
&= \lim_{h\to 0} \left[\cos(x)\,\frac{\cos(h)-1}{h}\right] - \lim_{h\to 0}\left[\sin(x)\,\frac{\sin(h)}{h}\right] && \text{by LL1} \\
&= \cos(x)\,\lim_{h\to 0} \frac{\cos(h)-1}{h} - \sin(x)\,\lim_{h\to 0}\frac{\sin(h)}{h} && \text{by LL3} \\
&= \cos(x)\cdot 0 - \sin(x)\cdot 1 \\
&= -\sin(x)
\end{aligned}$$

To summarize these results:

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x).$$

This results in a repeating pattern among the higher derivatives of sine and cosine:

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \cos(x)\\
\frac{d^2}{dx^2}\sin(x) &= \frac{d}{dx}\cos(x) = -\sin(x)\\
\frac{d^3}{dx^3}\sin(x) &= \frac{d}{dx}[-\sin(x)] = -\frac{d}{dx}\sin(x) = -\cos(x)\\
\frac{d^4}{dx^4}\sin(x) &= \frac{d}{dx}[-\cos(x)] = -\frac{d}{dx}\cos(x) = \sin(x)\\
\frac{d^5}{dx^5}\sin(x) &= \frac{d}{dx}\sin(x) = \cos(x)\\
&\vdots
\end{aligned}$$

And so on. The cycle repeats with a period of 4. The pattern is the same for the cosine function:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= -\sin(x)\\
\frac{d^2}{dx^2}\cos(x) &= \frac{d}{dx}[-\sin(x)] = -\cos(x)\\
\frac{d^3}{dx^3}\cos(x) &= \frac{d}{dx}[-\cos(x)] = -\frac{d}{dx}\cos(x) = \sin(x)\\
\frac{d^4}{dx^4}\cos(x) &= \frac{d}{dx}\sin(x) = \cos(x)\\
\frac{d^5}{dx^5}\cos(x) &= \frac{d}{dx}\cos(x) = -\sin(x)\\
&\vdots
\end{aligned}$$

Examples 9 and 10 show you two applications of derivatives. They are quite self-explanatory. Example 9 is similar to other examples we have looked at where we had a function $f(t)$ that gave the position of a moving object as a function of time. Example 10 demonstrates how the derivative can be interpreted in a business/economics context.