Notes for Section 2.4

It is time to learn how to differentiate products and quotients of functions. First of all,

$$(fg)' \neq f'g' \quad \text{and} \quad \left(\frac{f}{g}\right)' \neq \frac{f'}{g'}.$$

Here is the rule that actually works for products of functions.

Theorem (The Product Rule)**.** *Let f and g be functions of real numbers and $a \in \mathbb{R}$. If f and g are both differentiable at x, then fg is also differentiable at x and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

You can also express the Product Rule as

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

or

$$(fg)' = f'g + fg'.$$

I do not expect that you will have much difficulty in learning how to use these. Take a look at Examples 1-3 in Section 2.4 and let me know if you have questions. Here is an example for how the Product Rule can be used to differentiate a product of three functions:

Example. *Let us differentiate $f(x) = x^5 \sin(x)\cos(x)$.*

$$\begin{aligned}
\frac{d}{dx}[x^5 \sin(x)\cos(x)] &= \frac{d}{dx}\big(x^5[\sin(x)\cos(x)]\big) \\
&= \left[\frac{d}{dx}x^5\right][\sin(x)\cos(x)] + x^5\frac{d}{dx}[\sin(x)\cos(x)] \\
&= 5x^4\sin(x)\cos(x) + x^5\left(\left[\frac{d}{dx}\sin(x)\right]\cos(x) + \sin(x)\frac{d}{dx}\cos(x)\right) \\
&= 5x^4\sin(x)\cos(x) + x^5\left(\cos(x)\cos(x) + \sin(x)(-\sin(x))\right) \\
&= 5x^4\sin(x)\cos(x) + x^5(\cos^2(x) - \sin^2(x))
\end{aligned}$$

Now, you can try to differentiate $f(x) = x^5\sin(x)\cos(x)$ a different way for practice. Choose $g(x) = x^5\sin(x)$ and $h(x) = \cos(x)$ and use the Product Rule to differentiate $g(x)h(x)$ first, then use the Product Rule again to tackle $\frac{dg}{dx}$.

We owe the proof of the Product Rule. The one given in the book is fine, but not well motivated. Let me give a similar argument, which is somewhat longer, but perhaps easier to understand. So let f and g be functions that are differentiable at x. Let us start off by writing

$$\Delta f = f(x+h) - f(x) \qquad \text{and} \qquad \Delta g = g(x+h) - g(x).$$

So

$$f'(x) = \lim_{h\to 0}\frac{\Delta f}{h} \qquad \text{and} \qquad g'(x) = \lim_{h\to 0}\frac{\Delta g}{h}.$$

Now

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

If we substitute

$$f(x+h) = f(x) + \Delta f \qquad \text{and} \qquad g(x+h) = g(x) + \Delta g,$$

we get

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{\big(f(x)+\Delta f\big)\big(g(x)+\Delta g\big) - f(x)g(x)}{h}.$$

Let us work on the expression in the numerator a bit:

$$\begin{aligned}\big(f(x)+\Delta f\big)\big(g(x)+\Delta g\big)-f(x)g(x) &= f(x)g(x)+f(x)\Delta g+g(x)\Delta f+\Delta f\Delta g-f(x)g(x)\\ &= f(x)\Delta g+g(x)\Delta f+\Delta f\Delta g\end{aligned}$$

So

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x)\Delta g+g(x)\Delta f+\Delta f\Delta g}{h}\\ &= \lim_{h\to 0}\frac{f(x)\Delta g}{h}+\lim_{h\to 0}\frac{g(x)\Delta f}{h}+\lim_{h\to 0}\frac{\Delta f\Delta g}{h}\end{aligned}$$

by LL1. Now, $f(x)$ and $g(x)$ do not depend on h, so they are constant factors. Hence

$$\lim_{h\to 0}\frac{f(x)\Delta g}{h}=f(x)\lim_{h\to 0}\frac{\Delta g}{h}=f(x)g'(x),$$

and

$$\lim_{h\to 0}\frac{g(x)\Delta f}{h}=g(x)\lim_{h\to 0}\frac{\Delta f}{h}=g(x)f'(x)$$

by LL3. By LL4,

$$\lim_{h\to 0}\frac{\Delta f\Delta g}{h}=\left[\lim_{h\to 0}\frac{\Delta f}{h}\right]\left[\lim_{h\to 0}\Delta g\right]=f'(x)\lim_{h\to 0}\Delta g.$$

Now

$$\lim_{h\to 0}\Delta g=\lim_{h\to 0}[g(x+h)-g(x)]=\left[\lim_{h\to 0}g(x+h)\right]-\left[\lim_{h\to 0}g(x)\right]=\left[\lim_{h\to 0}g(x+h)\right]-g(x)$$

by LL2 and LL7 as $g(x)$ does not depend on h. Since g is differentiable at x, it is continuous at x by Theorem 4 in Section 2.2. So we can use direct substitution in

$$\lim_{h\to 0}g(x+h)=g(x+0)=g(x).$$

Hence

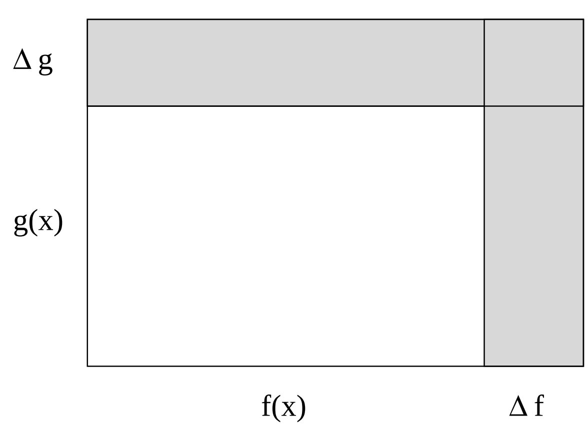
$$\lim_{h\to 0}\Delta g=g(x)-g(x)=0.$$

Let us put the pieces together:

$$\frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)+f'(x)\cdot 0=f(x)g'(x)+f'(x)g(x),$$

which is exactly the Product Rule.

What we just did here can be visualized using some geometry. Suppose for now that $f(x)$, $g(x)$, Δf, and Δg are all positive. Then $f(x+h)g(x+h)-f(x)g(x)$ is the area of the shaded L-shaped region in the following diagram:



As $h \to 0$, the two long rectangles become really skinny, while the rectangle that represents $\Delta f \Delta g$ becomes tiny even relative to those two skinny rectangles because both its width and height approach 0. This is why the numerator $f(x+h)g(x+h) - f(x)g(x)$ becomes $f(x)\Delta g + g(x)\Delta f$ in the limit, and this is why the Product Rule says $(fg)' = f'g + fg'$ and not $(fg)' = f'g'$.

Let us now deal with quotients.

Theorem (The Quotient Rule)**.** *Let f and g be functions of real numbers and $a \in \mathbb{R}$. If f and g are both differentiable at x and $g(x) \neq 0$, then f/g is also differentiable at x and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

You can also express the Quotient Rule as

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$$

or

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Skip the proof for now, and study Examples 4 and 5 in the book. Once you are comfortable with how the Quotient Rule is used in those examples, make up an example of a quotient from functions you know how to differentiate (i.e. power functions, root functions, sin and cos) and differentiate it. Note that unlike the Product Rule, the Quotient Rule is not symmetric in f and g because while $fg = gf$, $\frac{f}{g} \neq \frac{g}{f}$.

The Quotient Rule can be proved using a similar, but slightly more complicated argument as the Product Rule. The textbook has that proof, although once again I find the idea of adding and subtracting $f(x)g(x)$ quite unmotivated. I will give you my version of that proof at the end. Once again, my version is a little longer, but I think it is better motivated. Now, let me present an easier argument. Suppose g is a function of real numbers that is differentiable at $a \in \mathbb{R}$ and $g(a) \neq 0$. We will show that

$$\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{[g(a)]^2}$$

Notice that this is exactly the result you would get if you used the Quotient Rule to differentiate $1/g$ (try it!), but we are of course not allowed to use the Quotient Rule in its own proof because that would be circular reasoning. So we start by using the definition of the derivative:

$$\left(\frac{1}{g}\right)'(a) = \lim_{x \to a} \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x - a}.$$

Let us work on the numerator first:

$$\frac{1}{g(x)} - \frac{1}{g(a)} = \frac{g(a)}{g(x)g(a)} - \frac{g(x)}{g(x)g(a)} = \frac{g(a) - g(x)}{g(x)g(a)}$$

Substituting this back into the difference quotient:

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{x \to a} \frac{\frac{g(a)-g(x)}{g(x)g(a)}}{x - a} \\
&= \lim_{x \to a} \left[\frac{g(a) - g(x)}{x - a} \, \frac{1}{g(x)g(a)}\right] \\
&= \left[\lim_{x \to a} \frac{g(a) - g(x)}{x - a}\right] \left[\lim_{x \to a} \frac{1}{g(x)g(a)}\right] && \text{by LL4.}
\end{aligned}$$

Now,

$$\begin{aligned}\lim_{x\to a}\frac{g(a)-g(x)}{x-a} &= \lim_{x\to a}\left[-\frac{g(x)-g(a)}{x-a}\right] \\ &= -\lim_{x\to a}\frac{g(x)-g(a)}{x-a} && \text{by LL3} \\ &= -g'(a).\end{aligned}$$

In the other limit, first notice that $g(a)$ does not depend on x, so it is a constant factor. Hence

$$\lim_{x\to a}\frac{1}{g(x)g(a)} = \lim_{x\to a}\left[\frac{1}{g(x)}\,\frac{1}{g(a)}\right] = \frac{1}{g(a)}\lim_{x\to a}\frac{1}{g(x)}$$

by LL3. Since g is differentiable at a, it also continuous. Hence

$$\lim_{x\to a}\frac{1}{g(x)} = \frac{\lim_{x\to a}1}{\lim_{x\to a}g(x)} = \frac{1}{g(a)}$$

by LL5, then LL7 and the continuity of g at a. We can now put the pieces together:

$$\left(\frac{1}{g}\right)'(a) = -g'(a)\,\frac{1}{g(a)}\,\frac{1}{g(a)} = -\frac{g'(a)}{[g(a)]^2},$$

which is exactly what we wanted to show.

We can now combine this with the Product Rule to get the Quotient Rule:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \left(f\,\frac{1}{g}\right)'(x) \\ &= f'(x)\,\frac{1}{g(x)} + f(x)\left(\frac{1}{g}\right)'(x) \\ &= f'(x)\,\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{[g(x)]^2}\right) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} \\ &= \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}\end{aligned}$$

This is still fairly involved and quite a lengthy argument, but the individual steps are fairly straightforward and should look familiar from various limit calculations we have done in the past. In the next section, I will show you a much easier way to prove and even remember the Quotient Rule.

Now that we have the Quotient Rule, we can easily find the derivatives of tan, sec, csc, and cot using

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\ \frac{d}{dx}\sec(x) &= \frac{d}{dx}\frac{1}{\cos(x)} \\ \frac{d}{dx}\csc(x) &= \frac{d}{dx}\frac{1}{\sin(x)} \\ \frac{d}{dx}\cot(x) &= \frac{d}{dx}\frac{\cos(x)}{\sin(x)}.\end{aligned}$$

These are easy enough to do and I will leave the details to you. The book does the first one, but I'd say you should try doing it yourself instead of reading it. The results are given in the book, you can check your work against those.

Finally, here is my version of the argument the book uses to prove the Quotient Rule. Let f and g be functions of real numbers both differentiable at x such that $g(x) \neq 0$. Just like in the proof of the Product Rule, let

$$\Delta f = f(x+h) - f(x) \qquad \text{and} \qquad \Delta g = g(x+h) - g(x).$$

It is still true that

$$f(x+h) = f(x) + \Delta f \qquad \text{and} \qquad g(x+h) = g(x) + \Delta g$$

and

$$f'(x) = \lim_{h\to 0} \frac{\Delta f}{h} \qquad \text{and} \qquad g'(x) = \lim_{h\to 0} \frac{\Delta g}{h}.$$

Now

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}
\end{aligned}$$

Let us work on the expression in the numerator a bit:

$$\begin{aligned}
f(x+h)g(x) - f(x)g(x+h) &= \big(f(x) + \Delta f\big)g(x) - f(x)\big(g(x) + \Delta g\big) \\
&= f(x)g(x) + g(x)\Delta f - f(x)g(x) - f(x)\Delta g \\
&= g(x)\Delta f - f(x)\Delta g
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= \lim_{h\to 0} \frac{g(x)\Delta f - f(x)\Delta g}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{\frac{g(x)\Delta f - f(x)\Delta g}{h}}{g(x+h)g(x)} \\
&= \frac{\lim_{h\to 0} \frac{g(x)\Delta f - f(x)\Delta g}{h}}{\lim_{h\to 0}[g(x+h)g(x)]} && \text{by LL5}
\end{aligned}$$

Let us work on this one piece at a time. Since $g(x)$ does not depend on h,

$$\lim_{h\to 0}[g(x+h)g(x)] = g(x) \lim_{h\to 0} g(x+h)$$

by LL3. We know that g is differentiable at x, and so it is continuous at x. Hence

$$\lim_{h\to 0} g(x+h) = g(x+0) = g(x).$$

by direct substitution. Putting these last few pieces together,

$$\lim_{h\to 0}[g(x+h)g(x)] = g(x)g(x) = g^2(x).$$

Just like in our proof of the Product Rule,

$$\lim_{h\to 0} \frac{g(x)\Delta f}{h} = g(x) \lim_{h\to 0} \frac{\Delta f}{h} = g(x)f'(x)$$

and

$$\lim_{h\to 0} \frac{f(x)\Delta g}{h} = f(x) \lim_{h\to 0} \frac{\Delta g}{h} = f(x)g'(x).$$

Therefore

$$\lim_{h\to 0}\left(\frac{g(x)\Delta f}{h} - \frac{f(x)\Delta g}{h}\right) = \lim_{h\to 0}\frac{g(x)\Delta f}{h} - \lim_{h\to 0}\frac{f(x)\Delta g}{h} \qquad \text{by LL2}$$
$$= g(x)f'(x) - f(x)g'(x).$$

At last, putting all of the pieces together, we get

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$