Notes for Section 3.3

This section summarizes a number of connections between the derivative of a function f and the shape of the graph of f. We have already talked about quite a few of these, while others expand on ideas that will likely look familiar to you. For example, we already know that if a function is increasing over an interval, then its derivative is positive there, and if it is decreasing, then its derivative is negative. The section starts off by proving that the converse is also true: if the derivative of a function f is positive (negative) over an interval then f must be increasing (decreasing) over that the interval. The book refers to this as the Increasing/Decreasing Test. The proof is an interesting but easy consequence of the MVT and is done exactly the same way as the proof of Theorem 5 in Section 3.2. (Remember that I said that the significance of the MVT is that it can be used in proving other results?) The argument is quite straightforward and I would do it exactly the same way the textbook does it, so I will not spell it out here.

The First Derivative Test–which also makes good intuitive sense if you give it some thought–finally gives us a tool to tell if a critical point is a local maximum, minimum, or neither. We already know (by Fermat's Theorem) that if f has a local extremum at c, then c must be a critical point of f. Using the First Derivative Test, we can tell which critical point is a local minimum, which is a local maximum, and which is neither. The First Derivative Test follows from Increasing/Decreasing Test. If c is a point where f changes from decreasing to increasing then f must"bottom out" at c, that is f has a local minimum at c. Similarly, if f changes from increasing to decreasing at c, then f must have a peak at c, that is f has a local maximum at c. In terms of the derivative f', this would mean that c is a local minimum if f' is negative on some open interval (a, c) and f' is positive on some open interval (c, b); and similarly, c is a local maximum if f' is positive on some open interval (a, c) and f' is negative on some open interval (c, b). While these arguments are not completely rigorous, they should give you a good understanding of why the First Derivative Test works.

We have already talked about the connection between the second derivative and concavity and observed that if the graph of a function f is concave up (down) over some interval, then f'' is positive (negative) there. The converse is also true: the sign of f'' over an interval tells you if the graph of f is concave up or down over that interval. The textbook refers to this as the Concavity Test. It is possible (and not difficult) to prove this rigorously. The proof is in Appendix C. We could prove this in class but it seems to me that an intuitive understanding of what the sign of f'' tells us about whether f' increasing or decreasing and how an increasing/decreasing f' results in a graph that is concave down or up is more valuable to a calculus student than the actual proof. But be sure to gve some thought to the precise definition of the a concave up/down function in terms of the tangent line on p. 161.

This simple observation about concavity gives us another tool to tell if a critical point c such that $f'(c) = 0$ is a local maximum or minimum called the Second Derivative Test. What is says also makes good sense: if the graph of f is concave up, then we have a local minimum at c, and if it is concave down, then we have a local maximum. The book does not give a rigorous proof of this, and neither will we in class, but the test is used often, so you will want to be sure that you have a firm understanding of the intuitive reason behind it.

The Second Derivative Test is not quite as powerful as the First Derivative Test because

- it only works for critical points of the kind $f'(c) = 0$ and not when $f'(c)$ does not exist,
- it only works reliably if f'' is continuous near the critical point c (for our purposes in MCS 119, this is likely to be true),
- it can easily turn out to be inconclusive if $f''(c)$ turns out to be 0.

Despite all these weaknesses, the Second Derivative Test is often quicker or easier to do than the First Derivative Test.

Do not miss the definition of the inflection point. Next time the conversation at a party you are attending turns to inflection points, you do not want to be the only person in the room that does not have a good understanding of what an inflection point is.