# Notes for Section 3.7

Suppose I tell you that I thought of a function $f$ and its derivative is $f'(x) = 4x^3$. Can you guess what function I thought of? You would probably say "Easy, it was $f(x) = x^4$." You would almost be right. I could have thought of $f(x) = x^4 + 5$ or $f(x) = x^4 - \pi$ or any function of the form $f(x) = x^4 + c$ where $c$ is a constant number (your textbook uses a capital $C$, I like a lower case $c$). While we cannot tell exactly which of those functions I thought of, we can tell it must have been one of these. We know this because we learned in Section 3.2 (Corollary 7) that two functions that have the same derivative can only differ by a constant. Since we know $\frac{d}{dx}x^4 = 4x^3$, any other function whose derivative is $4x^3$ must look like $f(x) = x^4 + c$ for some constant $c$. Since we got these functions by thinking about reversing the derivative $f'(x) = 4x^3$, we say that they are *antiderivatives* of $4x^3$. So you can say $x^4$ is an antiderivative of $4x^3$ and $x^4 + 5$ is another antiderivative of $4x^3$. Since $f(x) = x^4 + c$ is the most general form, which includes all of the antiderivatives of $4x^3$, we call it the antiderivative of $4x^3$. While it is typically referred to as the antiderivative, $f(x) = x^4 + c$ is really a set of infinitely many functions. It is often called a family of functions to suggest that they are all related. The process of finding the antiderivative of a function is called antidifferentiation.

While it is unclear at this point why finding the antiderivative of a function may be useful, we will soon see that it allows us to deal with another challenge in calculus that has a lot of practical applications. For now, notice that what you know about derivatives allows you to find the antiderivatives of many functions. For example, if I tell you that $f'(x) = \cos(x)$, you can easily tell that $f(x) = \sin(x) + c$. We need to think a little harder if we want to figure out the antiderivative of $\sin(x)$. We know that $\frac{d}{dx}\cos(x) = -\sin(x)$. So a little intelligent guessing will lead us to $\frac{d}{dx}[-\cos(x)] = -[-\sin(x)]] = \sin(x)$. Therefore the antiderivative of $\sin(x)$ is $-\cos(x) + c$. Similar thinking allows us to figure out the antiderivative of $x^2$:

$$\frac{d}{dx}x^3 = 3x^2 \implies \frac{d}{dx}\left(\frac{1}{3}x^3\right) = \frac{1}{3}3x^2 = x^2$$

so the antiderivative of $x^2$ is $\frac{x^3}{3} + c$. By the same logic, the antiderivative of $x^n$ is $\frac{x^{n+1}}{n+1} + c$, except when $n = -1$, because

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{1}{n+1}\frac{d}{dx}x^{n+1} = \frac{1}{n+1}(n+1)x^n = x^n.$$

Notice that while finding the antiderivative involves a bit of intelligent guesswork, checking that our guess is correct is straightforward and involves no guesswork. I did not include the constant $c$ when differentiating because its derivative is 0 anyway, but there would be no harm if I did:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = \frac{1}{n+1}\frac{d}{dx}x^{n+1} + 0 = \frac{1}{n+1}(n+1)x^n = x^n.$$

Here is a table of antiderivatives, based on the derivatives we know. The first column is the original function $f$ and the second column is the most general antiderivative $F$, so $F' = f$.

| $f$ | $F$ |
|---|---|
| $x^n$ if $n \neq -1$ | $\frac{x^{n+1}}{n+1} + c$ |
| $\sin(x)$ | $-\cos(x) + c$ |
| $\cos(x)$ | $\sin(x) + c$ |
| $\sec^2(x)$ | $\tan(x) + c$ |
| $\csc^2(x)$ | $-\cot(x) + c$ |
| $\sec(x)\tan(x)$ | $\sec(x) + c$ |
| $\csc(x)\cot(x)$ | $-\csc(x) + c$ |

We can easily deal with sums, differences, and constant multiples of functions. Because of the simple forms of the Sum Rule, Difference Rule, and Constant Multiple Rule, these rules work backward just as easily as forward. For example, if we want to find the antiderivative of $7\sqrt[3]{x}$, we can say that the antiderivative of $\sqrt[3]{x} = x^{1/3}$ is $\frac{x^{4/3}}{4/3} + c = \frac{3}{4}\,x^{4/3} + c$, so the antiderivative of $7\sqrt[3]{x}$ is $7\,\frac{3}{4}\,x^{4/3} + c = \frac{21}{4}\,x^{4/3} + c$. In fact,

$$\frac{d}{dx}\left(\frac{21}{4}\,x^{4/3} + c\right) = \frac{21}{4}\frac{d}{dx}x^{4/3} = \frac{21}{4}\frac{4}{3}\,x^{1/3} = 7\sqrt[3]{x}.$$

You might feel that since the antiderivative of $\sqrt[3]{x}$ is $\frac{3}{4}\,x^{4/3} + c$, the antiderivative of $7x^3$ should be $7\left(\frac{3}{4}\,x^{4/3} + c\right) = \frac{21}{4}\,x^{4/3} + 7c$. That is actually correct. But since $c$ is any real number anyway, so is $7c$. And if we just want to say $\frac{21}{4}\,x^{4/3}$ plus any real number, we might as well write it in the simpler form as $\frac{21}{4}\,x^{4/3} + c$.

Let us try a more complicated example. We will find the antiderivative of $7\sqrt[3]{x} - 5\cos(x) + 4$. We can work on each term by itself. As we already noted, the antiderivative of $7\sqrt[3]{x}$ is $\frac{21}{4}\,x^{4/3} + c$. The antiderivative of $\cos(x)$ is $\sin(x) + c$, so the antiderivative of $5\cos(x)$ should be $5\sin(x) + c$. What about the antiderivative of 4? It is easy to see that $4x + c$ will work. In fact, this does follow from the table of antiderivatives above. We can write 4 as $4x^0$, and therefore its antiderivative is $4\,\frac{x^1}{1} + c = 4x + c$. So the antiderivative of $7\sqrt[3]{x} - 5\cos(x) + 4$ should be $7\,\frac{3}{4}\,x^{4/3} - 5\sin(x) + 4x + c$. It is eeasy enough to check that this is correct:

$$\frac{d}{dx}\left(\frac{21}{4}\,x^{4/3} - 5\sin(x) + 4x\right) = \frac{21}{4}\frac{4}{3}\,x^{1/3} - 5\cos(x) + 4 = 7\sqrt[3]{x} - 5\cos(x) + 4.$$

Notice that some of the functions in our table of antiderivatives are quite peculiar, such as $\csc(x)\cot(x)$. How likely are we to encounter exactly that function? On the other hand, some obvious ones are missing. For example, we do not have the antiderivative of $\tan(x)$. You might be tempted to think that since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, its antiderivative should be $\frac{-\cos(x)}{\sin(x)} + c$. But that would definitely be wrong. Because of the Quotient Rule, the derivative of $f/g$ is not $f'/g'$, and therefore the antiderivative of $f'/g'$ is not $f/g + c$ either. In fact,

$$\begin{aligned}
\frac{d}{dx}\frac{-\cos(x)}{\sin(x)} &= -\frac{d}{dx}\frac{\cos(x)}{\sin(x)} \\
&= -\frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{\sin(x)\sin(x) + \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{1}{\sin^2(x)} \\
&= \csc^2(x)
\end{aligned}$$

and not $\tan(x)$. Similarly, the antiderivative of $x\cos(x)$ is not $\frac{x^2}{2}\sin(x) + c$ because of the Product Rule, and the antiderivative of $\sin(x^2)$ is not $-\cos(x^2) + c$ and not even $-\cos\left(\frac{x^3}{3}\right) + c$ because of the Chain Rule.

In general, finding the antiderivatives of products, quotients, and composite functions is somewhat tricky and we do not yet know the tricks. Those tricks are mostly covered in Calculus 2. In fact, antidifferentiation is quite a bit more complex than differentiation. While the properties of derivatives we have learned (Sum Rule, Difference Rule, Constant Multiple Rule, Product Rule, Quotient Rule, Chain Rule, derivatives of power functions and trigonometric functions) allow us to find the derivative of just about any function made up of power funtion, root, and trigonometric function components, finding the antiderivatives of such complex functions is quite an art with many tricks, and the antiderivative often cannot even be expressed in terms of functions we are familiar with.

So what about higher antiderivatives? The same way we can differentiate a function several times to find its higher derivatives, we can also antidifferentiate several times to find its higher antiderivatives. Here is a quick example. What can we say about $f$ if $f''(x) = 3\sin(x)$? First, we can say that $f'(x) = -3\cos(x) + c$. Now we can say that $f(x) = -3\sin(x) + cx + d$ where $c$ and $d$ are any real numbers. Note how we got the $cx$ from the $c$ and we needed a new constant $d$. We had to assign a different letter to that new constant because there is no reason to believe it must have the same value as the constant $c$ that was already in the first antiderivative. You can differentiate $-3\sin(x) + cx + d$ twice to verify that its second derivative is indeed $3\sin(x)$.

Examples 5 and 6 about objects moving along a straight line (or in general along a fixed path that allows only forward or backward motion) give a hint about why antidifferentiation is a useful thing. They reverse the ideas about position and velocity that led us to derivatives. They illustrate how we can find the position of a moving object if we know its velocity and initial position, or find the velocity and position of the object from its acceleration. The problem of recovering position from velocity or acceleration has very definite practical applications. Prior to the availability of GPS, commercial and military aircraft, cruise missiles, submarines, and torpedoes all relied on this method to calculate their own positions. Such navigation systems are called inertial navigation systems and work by carefully measuring the velocity of the moving vehicle and calculating its position from this data. They are still in use at least as backup systems to GPS, or in places where GPS is not available, such as under the sea or in space. And calculating position from acceleration is how your smart phone or tablet knows how to orient its screen as you rotate it around.