MCS 119 Review Sheet

Here is a list of topics we have covered so far. Your review should certainly include your homework problems, as some of these will show up on the exam. You should also review the problems in your Webwork homework as the problem solving strategies you practiced while working on those may prove useful in solving problems on the exam.

The word "understand" is often used below. The definition of understanding is that you understand something when you know why it is true and can give a coherent and correct argument (proof) to convince someone else that it is true.

- Continuity
  - Definition of f being continuous at a in terms of the limit. Understand how this is the same as the Direct Substitution Property.
  - Types of discontinuities: removable, jump, and infinite discontinuities.
  - Left and right continuity, definitions, examples.
  - Continuity over an interval.
  - Understand why polynomials and rational functions are continuous at any number x in their domains.
  - Adding and subtracting continuous functions, multiplying by a scalar, multiplying, and dividing continuous functions.
  - Polynomials, rational functions, root functions, and trigonometric functions are continuous at any number x in their domains.
  - The limit of a composite function: if $\lim_{x\to a} g(x) = b$ and f is continuous at b, then $\lim_{x\to a} f(g(x)) = f(b)$. Proof?
  - If g is continuous at a, then $\lim_{x\to a} f(g(x)) = f\big(\lim_{x\to a} g(x)\big)$. The composition of two continuous functions is continuous.
  - The Intermediate Value Theorem, examples and applications.
- Infinite limits and limits at infinity
  - Informal definitions of
    - $\lim_{x\to a} f(x) = \infty$
    - $\lim_{x\to a} f(x) = -\infty$
    - $\lim_{x\to \infty} f(x) = L$
    - $\lim_{x\to -\infty} f(x) = L$
    - $\lim_{x\to \infty} f(x) = \infty$
    - $\lim_{x\to -\infty} f(x) = \infty$
    - $\lim_{x\to \infty} f(x) = -\infty$
    - $\lim_{x\to -\infty} f(x) = -\infty$

    and corresponding one-sided limits. Notice there are many of these, so rather than memorizing definitions, find a good system to be able construct the definition for yourself. Understanding why the definition says what it says is key.
  - Intuitive understanding of infinite limits based on what the values of a function do and what the graph looks like
  - A few computational tricks, such as $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ and $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ when p and q are polynomials.
  - Vertical and horizontal asymptotes, definitions and how to find them
  - Formal definitions of infinite limits with M, N, δ, ϵ. Again, there are are many combinations, especially if you include one-sided limits, so find a good system to be able to reconstruct these in your mind instead of memorizing. Proving what the limit is using the formal definition.

- Rates of change
  - The slope of a secant line and average rate of change. What it means, proper units, and how to calculate it using the difference quotient. Equation of the secant line.
  - The slope of the tangent line and instantaneous rate of change. The tangent line at $x = x_1$ is the limit of the secant lines between x_1 and x_2 as $x_2 \to x_1$. Physical units. Equation of the tangent line.
  - Definition of the derivative:
    * $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
    * $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

    Using the above definitions to find the slope of f. The limit can often be evaluated by the computational tools we have for working with limits: factoring, rationalizing roots, direct substitution into continuous functions, and the limit laws.
  - Various notations for the derivative: f', $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$.
  - The derivative as a function. The connection between the graph of f and the graph of f'. Understand why $f > 0$ if f is increasing, $f' < 0$ if f is decreasing, and $f' = 0$ if the tangent line is horizontal.
  - Differentiability of f. If f is differentiable at x then f must be continuous at x. Proof.
  - The reasons why $f'(x)$ may not exist. Some of the most typical reasons are: f is discontinuous, f has a vertical tangent line, the graph of f has a sharp corner (cusp). Understand why these make the limit of the difference quotient not exist. Note that these are not the only reasons that $f'(x)$ may not exist.
  - Higher derivatives. Notation $f''(x) = \dfrac{d^2 f}{dx^2}$, $f^{(n)}(x) = \dfrac{d^n f}{dx^n}$, etc.
  - f'' as the rate of change of f'. Roughly speaking, f'' is the acceleration of f. Understand why
    * $f'' > 0$ if f' is increasing, which happens if the graph of f is curving up,
    * $f'' < 0$ if f' is decreasing, which happens if the graph of f is curving down.
  - Basic differentiation formulas:
    * Constant function: $\dfrac{d}{dx}c = 0$. Intuition: f is constant, its value does not change, so its rate of change is 0. Proof using the definition of the derivative.
    * Power function: $\dfrac{d}{dx}x^n = nx^{n-1}$. Holds for any $n \in \mathbb{R}$. Proof for $n \in \mathbb{Z}^+$ using the definition of the derivative.
    * Constant multiple: $\dfrac{d}{dx}cf(x) = c\dfrac{d}{dx}f(x)$ if f is differentiable at x. Interpretation in terms of vertically stretching/compressing the graph of f. Proof using the definition of the derivative.
    * Sum of functions: $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{df}{dx} + dgdx$ if f and g are differentiable at x. Proof using the definition of the derivative.
    * Difference of functions: $\dfrac{d}{dx}[f(x) - g(x)] = \dfrac{df}{dx} - dgdx$ if f and g are differentiable at x. Proof using the definition of the derivative.
    * The product rule: $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$ if f and g are differentiable at x. Proof using the definition of the derivative.
    * The quotient rule: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ if f and g are differentiable at x. Proof using the product rule and the chain rule.

    - ∗ The chain rule: $\frac{d}{dx} f \circ g(x) = f'\big(g(x)\big)\, g'(x)$ if f is differentiable at $g(x)$ and g is differentiable at x. Somewhat defective proof by using the definition of the derivative and multiplying by $\frac{g(x)-g(a)}{g(x)-g(a)}$ or by $\frac{g(x+h)-g(x)}{g(x+h)-g(x)}$. Understand why this proof is not quite correct.
    - ∗ The derivatives of trigonometric functions:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

      Proof of $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$ using approrpiate trig identities and the unit circle. Proofs of the other four by using the quotient rule.
  - Implicit differentiation. Understand how you can find the slope of a curve given by an equation in x and y by differentiating with respect to x while treating $y = y(x)$ as an implicit function of x in a neighborhood of a point on the curve. Know how to find the equation of the tangent line and higher derivatives. Understand why a curve may not have a slope at a point (e.g. vertical tangent line, self-intersection).
  - Related rates. Understand how the rates of change of several quantities that are related by an equation are related to each other. Know how to use the chain rule to solve word problems about related rates.

- Applications of differentiation
  - Definitions of local and absolute extrema and critical points. Fermat's Theorem and its proof. The Extreme Value Theorem. The closed interval method for finding the absolute extrema of a continuous function over a closed interval $[a, b]$.
  - Rolle's Theorem. Proof using the Extreme Value Theorem and Fermat's Theorem.
  - The Mean Value Theorem. Proof using Rolle's Theorem. Understand why the function needs to be continuous on $[a, b]$ and differentiable on (a, b).
  - Theorem that a function whose derivative on an interval is 0 must be a constant function. Proof using the MVT. Corollary that says if $f'(x) = g'(x)$ on some interval then $f(x) = g(x) + c$ on that interval and its proof.
  - The connection between f' and f. If $f'(x) > 0$ on some interval then f is increasing on that interval; and if $f'(x) < 0$ on some interval then f is decreasing on that interval. Proofs of these last two. The First Derivative Test to determine if a critical point is a local min or max or neither.
  - The connection between f'' and f. Definitions of concave up, concave down, inflection point. If $f''(x) > 0$ on some interval then f is concave up on that interval; and if $f'.(x) < 0$ on some interval then f is concave down on that interval. The Second Derivative Test to determine if a critical point is a local min or max.
  - Antiderivatives. Definition of antiderivative. The role of the constant c in the most general form of the antiderivative. Know how to use the rules of differentiation backwards to find the antiderivative of a function. Rectilinear motion: using antiderivatives to recover velocity from acceleration and position from velocity.

- The area under the graph
  - Approximating the value of a function from its derivative. The distance problem: approximating distance traveled by summing the product of velocity by time over short time intervals.

– Approximating the area under the graph of f by rectangles. The Riemann sum, partition of an interval, sample points. Special cases of the Riemann sum: left-hand sums, right-hand sums, the midpoint rule, lower sums, upper sums. Understand how to calculate a Riemann-sum approximation using a finite number of rectangles.
– Area under the graph defined as the limit of the Riemann sum. Calculating such a limit directly from the definition, using properties of sums and limits. Definition of the definite integral and integrable function. Integral notation, dummy variables. Understand why areas under the x-axis count as negative.
– Theorem: A function f that is continuous or has only a finite number of jump discontinuities on a closed interval $[a, b]$ is integrable on $[a, b]$.
– Understand how the definite integral can be found in some special cases using area formulas for familiar shapes and symmetry instead of calculating limits.
– Properties of the definite integral:
  * $\int_a^b c\,dx = c(b-a)$
  * $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  * $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
  * $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
  * If $0 \leq f(x)$ for all $x \in [a, b]$ then $0 \leq \int_a^b f(x)\,dx$
  * If $f(x) \geq g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
  * If $m \leq f(x) \leq M$ for all $x \in [a, b]$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$
– The definition of the indefinite integral of f as the most general antiderivative of f. Make sure you understand the difference between the definite and the indefinite integrals: the former is the area under the graph (that is a number), the latter is the antiderivative (that is a function).
– The Fundamental Theorem of Calculus establishes the connection between integration and differentiation: If f is a continuous function on the interval $[a, b]$ then
  1. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ for any $x \in [a, b]$.
  2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative (or indefinite integral) of f, that is $F'(x) = f(x)$.