MCS 119 Exam 1

All of your answers must be carefully justified. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. You may refer to any result proved in class unless otherwise specified. You may use results you proved on your homework, except for ones the problem specifically asks you to prove.

You are not allowed to use your textbook or your class notes, but you may use a simple calculator.

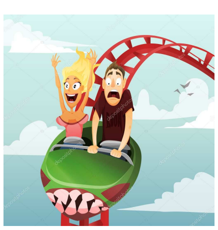
1. (10 pts) Calculate the exact value of

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}.$$

Make sure you carefully justify the steps in your calculation.

2. (5 pts each) Let f be a function of real numbers and let a be a real number. State the informal and formal definitions of the following.
   (a) $\lim\limits_{x\to a^-} f(x) = \infty$
   (b) $\lim\limits_{x\to\infty} f(x) = -\infty$

3. (5 pts each) Let f be a function of real numbers such that f is continuous on the interval $[2, 6]$, $f(2) = -1$, and $f(6) = 1$.
   (a) Below is an argument to show that there is a number c such that the value of $g(x) = 1/f(x)$ at $x = c$ is $1/2$. The argument has a mistake in it. Find it and explain why it is a mistake.
   Let $g(x) = 1/f(x)$. We know $f(2) = -1$ and $f(6) = 1$, therefore $g(2) = -1$ and $g(6) = 1$. Since f is continuous on the closed interval $[2, 6]$, $g(x) = 1/f(x)$ is also continuous. The number $1/2$ is between -1 and 1, hence there must be a number $c \in (2, 6)$ such that $g(c) = 1/2$ by the Intermediate Value Theorem. (Hint: You may find answering part (b) first easier. Or perhaps answering part (a) will lead you to the right idea for part (b). It all depends on how you think about it. But read both questions and see which one gives you an idea.)
   (b) Find an example of a function f that satisfies the conditions of this problem, yet there is no number $c \in (2, 6)$ such that that the value of $g(x) = 1/f(x)$ at $x = c$ is $1/2$. Justify your example.



4. Superman: Escape from Krypton is a vertical roller coaster at the Six Flags Magic Mountain amusement park just outside the Los Angeles area. The coaster has a roughly L-shape track, except of course the part between the horizontal and the vertical parts is rounded, not sharp like the vertex in the letter L. The train takes off horizontally, accelerates, then quickly turns vertical and shoots up while decelerating under the force of gravity. It seems to stop for a split moment at the top of the track before falling back down and eventually returning to the station.
Suppose that

$$h(t) = -5t^2 + 45t + 25$$

is the height of the train at time t during the vertical phase of the ride, where h is measured in meters and t is in seconds.

   (a) (9 pts) Use the definition of the derivative (either form) to find the (vertical) velocity of the train at $t = a$.

(b) (1 pt) Just for fun, how fast is the train moving when it starts it ascent at $t = 0$? Does this seem fast to you?

5. **Extra credit problem.** The goal of this problem is to find the derivative of $f(x) = \sqrt[4]{x}$ at $a > 0$. In class, we used the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$ at $a > 0$ by expressing it as

$$f'(a) = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a},$$

then multiplying the numerator and the denominator by the conjugate $\sqrt{x} + \sqrt{a}$.

We now want to use the definition of the derivative to find the derivative of $f(x) = \sqrt[4]{x}$ at $a > 0$. We can write

$$f'(a) = \lim_{x \to a} \frac{\sqrt[4]{x} - \sqrt[4]{a}}{x - a}.$$

(a) (2 pts) Try using the same trick: multiply the numerator and the denominator by $\sqrt[4]{x} + \sqrt[4]{a}$ and simplify as much as possible.

(b) (2 pts) What you did in part (a) is not quite enough to evaluate the limit. But it should have resulted in an expression that looks familiar. Try the same trick again: multiply the numerator and the denominator by the conjugate and simplify as much as possible.

(c) (6 pts) Use what you know about limits to evaluate the limit you got in part (b). Make sure you carefully justify the steps of your calculation.