## MCS 119 FINAL EXAM

All of your answers must be carefully justified. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. You may refer to any result proved in class unless otherwise specified. You may use results you proved on your homework, except for ones the problem specifically asks you to prove.

You are not allowed to use your textbook or your class notes, but you may use a simple calculator.

1. (10 pts) Let

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}.$$

Find the horizontal asymptotes of f by calculating the exact values of the limits

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \text{ and } \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}.$$

Did you get the same value or different values for these two limits?

2. (10 pts) Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point  $(x_0, y_0)$  is the line with equation

$$\frac{x_0}{a^2} \, x + \frac{y_0}{b^2} \, y = 1$$

3. Let  $f : \mathbb{R} \to \mathbb{R}$  be the function

$$f(x) = \frac{x^2 - 4}{x^2 + 4}.$$

- (a) (8 pts) Find the intervals of concavity, that is those intervals on which f is concave up and those on which f is concave down.
- (b) (2 pts) What are the inflection points of f? How do you know these are inflection points?
- 4. (a) (4 pts) When asked to use the definition of the derivative to find the derivative of  $f(x) = \sqrt{x}$  at x = 1/2, Bart Simpson, who is not much of a calculus whiz, wrote the following:

$$f'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{\sqrt{1/2 + h} - \sqrt{1/2}}{h} = \frac{\sqrt{1/2} + \sqrt{h} - \sqrt{1/2}}{h} = \frac{\sqrt{h}}{h} = \frac{0}{0} = 1$$

Find all of Bart's mistakes and explain why they are wrong.

(b) (6 pts) Use the definition of the derivative (correctly) to find the derivative of  $f(x) = \sqrt{x}$  for any real number x > 0.



5. (10 pts) In recognition of his highly advanced engineering skills, Chewbacca is appointed to design a holding pen for the rebels' tauntauns on the planet Hoth. The holding pen is to be rectangular in shape, divided into three equal size rectangles by two rows of fencing parallel to one of the side, as in the diagram:



The reason the pen is to be divided into three parts has something to do with tauntaun biology. Chewbacca is not an expert at tauntaun biology, so instead of getting hung up on this detail, he would rather focus on designing the pen. He has 420 meters of fencing material available to him, and wants to build the holding pen with the largest total area possible. What is the largest possible area of such a holding pen? How do you know this is the absolute maximum?

- 6. (5 pts each)
  - (a) Let  $f(x) = x^2 + 2$ . Use a left-hand sum with a uniform partition of four subintervals to estimate the area under the graph of f from x = 1 to x = 3.
  - (b) Sketch a diagram of your estimate. Do you think the left-hand sum you calculated in part (a) underestimates or overestimates the area under the graph of f between x = 1 and x = 3? Why?
- 7. Extra credit problem. Let  $f : \mathbb{R} \to \mathbb{R}$  be the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

(a) (8 pts) Determine if f is continuous at x = 0 by finding the limit

$$\lim_{x \to 0} f(x)$$

if it exists or showing that it does not exist. (Hint: You may want to look at one-sided limits and use the Squeeze Theorem or a  $\delta - \epsilon$  argument.)

(b) (7 pts) Is f differentiable at 0? Why or why not?

