MCS 119 Exam 1 Solutions

1. (10 pts) Calculate the exact value of
$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}.$$
Do not forget to carefully justify the steps of your calculation.

Since $x\to\infty$, x can be treated as a large positive number, and so we do not need to be concerned about what happens when $x=0$. Therefore we can divide the numerator and the denominator by x:
$$\frac{\sqrt{2x^2+1}}{3x-5}=\frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}}=\frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{3-\frac{5}{x}}=\frac{\sqrt{\frac{2x^2+1}{x^2}}}{3-\frac{5}{x}}=\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}},$$
where we used the fact that $x=\sqrt{x^2}$ since x is positive. We know that
$$\lim_{x\to\infty}\frac{1}{x^2}=0,$$
and hence
$$\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)=2+0=2$$
by LL1 and LL7, and finally,
$$\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}=\sqrt{\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)}=\sqrt{2}$$
by LL11. Similarly,
$$\lim_{x\to\infty}\frac{1}{x}=0\implies\lim_{x\to\infty}\left(3-\frac{5}{x}\right)=\lim_{x\to\infty}3-5\lim_{x\to\infty}\frac{1}{x}=3-0=3$$
by LL2, LL3, and LL7. Therefore by LL5,
$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\frac{\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim_{x\to\infty}\left(3-\frac{5}{x}\right)}=\frac{\sqrt{2}}{3}.$$

Here is another way this can be argued, which is less rigorous but is still convincing enough. If x is a large number, then
$$2x^2+1\approx 2x^2\quad\text{and}\quad 3x-5\approx 3x.$$
The relative error (compared to the size of the number) in these estimates gets smaller and smaller as x gets larger because the $2x^2$ and the $3x$ dominate compared to the 1 and the -5. So the estimates are getting better and better as $x\to\infty$. In fact, the estimates are perfect in the limit. Therefore
$$\frac{\sqrt{2x^2+1}}{3x-5}\approx\frac{\sqrt{2x^2}}{3x}$$
when x is large. This estimate also gets better and better as $x\to\infty$ and is perfect in the limit. Hence
$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\lim_{x\to\infty}\frac{\sqrt{2x^2}}{3x}.$$
Since $x\to\infty$, we can treat x as a (large) positive number. So
$$\sqrt{2x^2}=\sqrt{2}\sqrt{x^2}=\sqrt{2}|x|=\sqrt{2}x,$$

and

$$\frac{\sqrt{2x^2}}{3x} = \frac{\sqrt{2}x}{3x} = \frac{\sqrt{2}}{3}$$

because we can cancel x as $x \neq 0$. Finally, by LL7,

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2}}{3} = \frac{\sqrt{2}}{3}$$

.

2. (5 pts each) The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars/pound is $Q = f(p)$.

   (a) What is the meaning of the derivative $f'(8)$? What are its units?

   The value of $f'(8)$ is the instantaneous rate of change of f at $p = 8$. It is approximately the change in the quantity of coffee sold if the price is increased by a dollar/pound from \$8/lb to \$9/lb. So the units are lb/(\$/lb) = lb^2/\$.

   (b) Is $f'(8)$ positive or negative? Explain.

   It is likely to be negative. The amount of coffee sold most likely decreases if the price increases, as people buy less coffee and less people buy coffee.

3. (10 pts) Use the definition of the derivative to differentiate $f(x) = x^5$.

   Hint: The $x \to a$ version of the definition results in a limit that is easier to tackle than the $h \to 0$ version.

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{x^5 - a^5}{x-a} \\
&= \lim_{x\to a} \frac{(x-a)(x^4 + x^3a + x^2a^2 + xa^3 + a^4)}{x-a} \\
&= \lim_{x\to a} (x^4 + x^3a + x^2a^2 + xa^3 + a^4) \\
&= a^4 + a^3a + a^2a^2 + aa^3 + a^4 \\
&= 5a^4,
\end{aligned}$$

where we could cancel $x - a$ because it is not 0 as $x \to a$ and we could use direct substitution to evaluate the last limit because $x^4 + x^3a + x^2a^2 + xa^3 + a^4$ is a polynomial in x. Hence the derivative is $f'(a) = 5a^4$, or with x as the variable, $f'(x) = 5x^4$

4. (10 pts) R2-D2 and Luke Skywalker are captured and taken to Darth Vader's star destroyer. They manage to escape from their prison cell, but getting off the ship is another challenge. Luke figures that with some luck, they can gain access to an escape pod, and if they eject the pod without activating its engine, there is a good chance they will not be detected. The pod drifting in space unpowered will appear to be space trash to the star destroyer's security sensors. R2 calculates that the star destroyer is flying along the curve $y = 5 + 4x - x^2$ from left to right, and there is a planet they could land and hide on at coordinates $(6, 2)$. Find the coordinates of the point along the star destroyer's path where the escape pod should be launched so that as it continues to fly in the direction of the tangent line, it will reach the planet at $(6, 2)$.

   Suppose the right coordinates to launch the escape pod are $(a, y(a)) = (a, 5 + 4a - a^2)$. The slope of the tangent line at this point is $y'(a) = 4 - 2a$. For the pod to reach the planet at $(6,2)$, the tangent line must pass through both $(a, 5+4a-a^2)$ and $(6,2)$. Hence its slope must be

$$m = \frac{5+4a-a^2-2}{a-6} = \frac{3+4a-a^2}{a-6}.$$

But a line can have only one slope, so these two slopes must be equal:

$$\begin{aligned}
\frac{3+4a-a^2}{a-6} &= 4-2a \\
3+4a-a^2 &= (4-2a)(a-6) && \text{for sure, } a-6 \text{ can't be } 0 \\
3+4a-a^2 &= 4a-24-2a^2+12a \\
a^2-12a+27 &= 0 \\
(a-3)(a-9) &= 0
\end{aligned}$$

Hence either $a=3$ or $a=9$. We know the destroyer travels along $y=5+4x-x^2$ from left to right, that is in the direction of increasing values of the x-coordinate. Hence the correct value is $x=3$ since it comes before 6, while $a=9$ would be after 6. The tangent line to $y=5+4x-x^2$ also passes through $(6,2)$, but the escape pod would be heading in the wrong direction, away from the planet at $(6,2)$, along this tangent line, if it were launched at $x=9$. So the coordinates of the point where the pod should be launched are $(3,8)$.

5. (10 pts)**Extra credit problem.** By the definition of the derivative,

$$\frac{d}{dx}\tan(x) = \lim_{h\to 0} \frac{\tan(x+h)-\tan(x)}{h}.$$

Use the trigonometric identity

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

to find the value of this limit.

Hint: Remember that we learned how to evaluate $\lim_{\theta\to 0} \frac{\tan(\theta)}{\theta}$ last semester.

Let us work on the difference quotient in the limit:

$$\begin{aligned}
\frac{\tan(x+h)-\tan(x)}{h} &= \frac{\frac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)} - \tan(x)}{h} \\
&= \frac{\frac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)} - \frac{\tan(x)-\tan^2(x)\tan(h)}{1-\tan(x)\tan(h)}}{h} \\
&= \frac{\tan(x)+\tan(h)-\tan(x)+\tan^2(x)\tan(h)}{h(1-\tan(x)\tan(h))} \\
&= \frac{\tan(h)+\tan^2(x)\tan(h)}{h(1-\tan(x)\tan(h))} \\
&= \frac{\tan(h)(1+\tan^2(x))}{h(1-\tan(x)\tan(h))} \\
&= \frac{\tan(h)}{h}\frac{1+\tan^2(x)}{1-\tan(x)\tan(h)}
\end{aligned}$$

So

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \lim_{h\to 0}\left[\frac{\tan(h)}{h}\frac{1+\tan^2(x)}{1-\tan(x)\tan(h)}\right]\\ &= \lim_{h\to 0}\frac{\tan(h)}{h}\cdot\lim_{h\to 0}\frac{1+\tan^2(x)}{1-\tan(x)\tan(h)}\end{aligned}$$

by Limit Law 4. We will deal with the limits in the two factors one at a time. First,

$$\begin{aligned}\lim_{h\to 0}\frac{\tan(h)}{h} &= \lim_{h\to 0}\frac{\frac{\sin(h)}{\cos(h)}}{h}\\ &= \lim_{h\to 0}\left[\frac{\sin(h)}{h}\frac{1}{\cos(h)}\right]\\ &= \underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1}\cdot\lim_{h\to 0}\frac{1}{\cos(h)} && \text{by LL4}\\ &= \frac{\lim_{h\to 0}1}{\lim_{h\to 0}\cos(h)} && \text{by LL5}\\ &= \frac{1}{1} && \text{by LL7 and continuity of cosine.}\end{aligned}$$

For the second factor, first note that

$$1+\tan^2(x) = 1+\frac{\sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x),$$

and this clearly does not depend on h. Hence

$$\begin{aligned}\lim_{h\to 0}\frac{1+\tan^2(x)}{1-\tan(x)\tan(h)} &= \lim_{h\to 0}\frac{\sec^2(x)}{1-\tan(x)\tan(h)}\\ &= \sec^2(x)\,\lim_{h\to 0}\frac{1}{1-\tan(x)\tan(h)} && \text{by LL3}\\ &= \sec^2(x)\,\frac{\lim_{h\to 0}1}{\lim_{h\to 0}(1-\tan(x)\tan(h))} && \text{by LL5}\\ &= \sec^2(x)\,\frac{1}{\lim_{h\to 0}1-\lim_{h\to 0}\tan(x)\tan(h)} && \text{by LL7 and LL2}\\ &= \sec^2(x)\,\frac{1}{1-\tan(x)\lim_{h\to 0}\tan(h)} && \text{by LL7 and LL3}\\ &= \sec^2(x)\,\frac{1}{1-\tan(x)\tan(0)} && \text{by continuity of tangent}\\ &= \sec^2(x)\,\frac{1}{1} && \text{by }\tan(0)=0\end{aligned}$$

Putting the pieces back together, we obtain the familiar result

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$