MCS 119 Exam 2 Solutions

1. (5 pts each)

(a) Write $|x| = \sqrt{x^2}$, and use the chain rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}.$$

$$\begin{aligned}
\frac{d}{dx}|x| &= \frac{d}{dx}\sqrt{x^2} \\
&= \frac{d}{dx}\left(x^2\right)^{1/2} \\
&= \frac{1}{2}\left(x^2\right)^{-1/2}\frac{d}{dx}x^2 \\
&= \frac{1}{2\sqrt{x^2}}\,2x \\
&= \frac{x}{|x|}
\end{aligned}$$

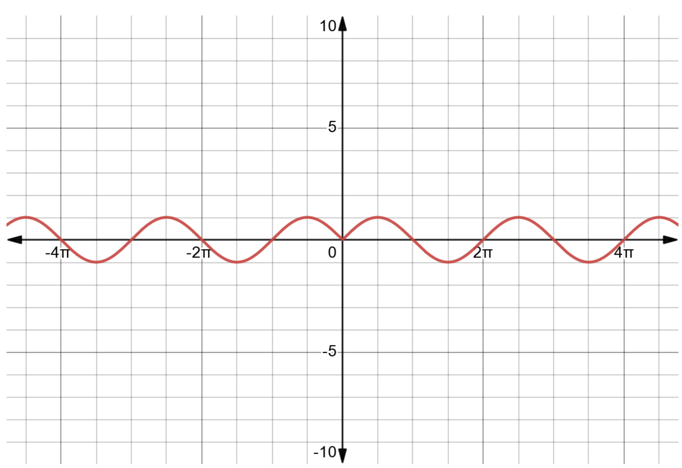
(b) If $g(x) = \sin|x|$, find $g'(x)$. Where is g not differentiable and why not? What does the graph of g look like at the point(s) where g is not differentiable?

$$\begin{aligned}
\frac{d}{dx}\sin(|x|) &= \cos(|x|)\frac{d}{dx}|x| \\
&= \cos(|x|)\,\frac{x}{|x|}
\end{aligned}$$

The one place where g' is suspect is at $x = 0$, since we end up with a 0 in the denominator. Everywhere else, $g'(x) = \cos(|x|)\,\frac{x}{|x|}$ has a legitimate value. In fact,

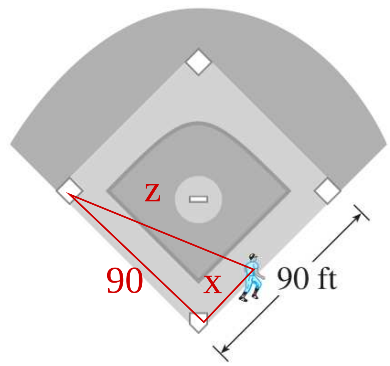
$$g(x) = \left\{\begin{matrix} \sin(x) & \text{if } x \geq 0 \\ \sin(-x) & \text{if } x < 0 \end{matrix}\right\} = \begin{cases} \sin(x) & \text{if } x \geq 0 \\ -\sin(x) & \text{if } x < 0. \end{cases}$$

So for $x \geq 0$, the graph of g is the same as the graph of the sine function. For $x < 0$, it is the graph of the sine function reflected across the x-axis:



So g has a cusp at $x = 0$, which is why it is not differentiable there.

2. (10 pts) A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s. At what rate is his distance from third base increasing when he is halfway to first base?



Consider the right triangle formed by third base, the batter's box, and the batter. The leg of this triangle between third base and batter's box has a constant length of 90 ft. Let

$x(t)$ = the distance between the batter and the batter's box at time t

$z(t)$ = the distance between the batter and third base at time t

Then

$$z^2 = x^2 + 90^2.$$

Differentiating this equation with respect to t gives

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 0 \implies z\frac{dz}{dt} = x\frac{dx}{dt}.$$

At the present time, we know $x = 45$ft and $\frac{dx}{dt} = 24$ft/s. The latter is positive because the batter's distance from the batter's base is increasing. We can find z by using the Pythagorean Theorem:

$$z^2 = 45^2 + 90^2 \implies z = \sqrt{45^2 + 90^2} = 45\sqrt{5}.$$

Hence

$$45\sqrt{5}\frac{dz}{dt} = 45(24) \implies \frac{dz}{dt} = \frac{24}{\sqrt{5}}.$$

So the batter's distance from second base is increasing at $24/\sqrt{5}$ ft/s ≈ 10.73 ft/s.

3. (10 pts) The curve $x^2 - 2y^2 = 4$ describes a hyperbola. There are two tangent lines to this curve that pass through the point $(2, 2)$. Use implicit differentiation to find the equations of these two tangent lines.

Differentiating both sides of the equation with respect to x gives

$$\frac{d}{dx}(x^2 - 2y^2) = \frac{d}{dx}4$$
$$2x - 4y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{2x}{4y} = \frac{x}{2y}.$$

Suppose the tangent line intersects the curve at the point (a, b), Then its slope there is

$$\frac{dy}{dx} = \frac{a}{2b}.$$

But the tangent line we are looking for also passes through $(2, 2)$, so its slope must be

$$\frac{\Delta y}{\Delta x} = \frac{b-2}{a-2}.$$

Since a line can have only one slope, these two must be equal:

$$\frac{a}{2b} = \frac{b-2}{a-2}.$$

If $b \neq 0$ and $a \neq 2$, this gives the quadratic equation

$$\begin{aligned} a(a-2) &= 2b(b-2) \\ a^2 - 2a &= 2b^2 - 4b \\ a^2 - 2b^2 - 2a + 4b &= 0 \end{aligned}$$

Remember that (a, b) is also on the hyperbola, and so it must satisfy $a^2 - 2b^2 = 4$. Hence can replace the $a^2 - 2b^2$ on the left-hand side above by 4:

$$4 - 2a + 4b = 0 \implies 2a = 4 + 4b \implies a = 2 + 2b.$$

We can now substitute this into $a^2 - 2b^2 = 4$:

$$\begin{aligned} (2+2b)^2 - 2b^2 &= 4 \\ 4 + 8b + 4b^2 - 2b^2 &= 4 \\ 2b^2 + 8b &= 0 \\ b^2 + 4b &= 0 \\ b(b+4) &= 0 \end{aligned}$$

Hence $b = 0$ or $b = -4$. The corresponding values of a are $a = 2$ or $a = -6$. So one tangent line passes through $(2, 2)$ and $(-6, -4)$, and so its equation is

$$y = \frac{2-(-4)}{2-(-6)}(x-2) + 2 \implies y = \frac{3}{4}x + 2.$$

The other tangent line passes through $(2, 2)$ and $(2, 0)$, and so it is a vertical line with equation $x = 2$.

Actually, there is a mistake in the solution above. Remember that at some point along the way we multiplied both sides of an equation by $2b$ and by $a-2$, while we assumed neither of these was 0. In fact, they both turned out to be 0 for the second tangent line. Interestingly, the tangent line and its equation are still correct. We can fix this mistake by differentiating with respect to y instead of x:

Differentiating both sides of the equation with respect to y gives

$$\begin{aligned} \frac{d}{dy}(x^2 - 2y^2) &= \frac{d}{dy}4 \\ 2x\frac{dx}{dy} - 4y &= 0 \\ \frac{dx}{dy} &= \frac{4y}{2x} = \frac{2y}{x}. \end{aligned}$$

At $(2, 0)$,

$$\frac{dx}{dy} = \frac{0}{2} = 0.$$

This tells us that the hyperbola has a vertical tangent line at $(2, 0)$, whose equation is $x = 2$. Sure enough, this line happens to pass through $(2, 2)$. We have indeed found a second tangent line through $(2, 2)$ without having to anything problematic, such as dividing by 0.

4. (10 pts) Let f be a function of real numbers and let $c \in \mathbb{R}$ be a local minimum of f. Prove that c must be a critical point. Note that this result is part of Fermat's Theorem, and so you cannot use Fermat's Theorem to prove it, as that would be circular reasoning.

   Hint: If $f'(c)$ exists, consider the left and right limits of the difference quotient to argue that the two-sided limit must be 0.

   Suppose f has a local minimum at $x = c$. Then $f(x) \geq f(c)$ for every x in some small enough neighborhood of c. Let us consider the right limit
   $$\lim_{x\to c^+} \frac{f(x) - f(c)}{x - c}.$$
   Once x is close enough to c (and $x > c$), we have
   $$f(x) \geq f(c) \implies f(x) - f(c) \geq 0 \implies \frac{f(x) - f(c)}{x - c} \geq 0$$
   for every $x > c$ close to c. Hence
   $$\lim_{x\to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$
   if the right limit exists at all.

   Similarly, let us look at the left limit
   $$\lim_{x\to c^-} \frac{f(x) - f(c)}{x - c}.$$
   Once x is close enough to c (and $x < c$), we have
   $$f(x) \geq f(c) \implies f(x) - f(c) \geq 0 \implies \frac{f(x) - f(c)}{x - c} \leq 0$$
   for every $x < c$ close to c. Hence
   $$\lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$
   assuming the left limit exists.

   Now, if f is differentiable at $x = c$, then
   $$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$
   exists. But this is only possible if both the left and right limits exist and are equal. So
   $$0 \leq \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} \leq 0.$$
   This is only possible if all four quantities in this inequality are 0. So
   $$0 = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

5. (5 pts each) **Extra credit problem.** Let n be a positive integer and $f(x) = \sqrt[n]{x}$. We remarked in class that f can be differentiated using the power rule as
   $$\frac{d}{dx}\sqrt[n]{x} = \frac{d}{dx}x^{1/n} = \frac{1}{n}\,x^{1/n-1}.$$

But we never proved that the power rule works for powers with fractional exponents. Here is your chance.

(a) First, note that $[f(x)]^n = \sqrt[n]{x}^n = x$ for every x in the domain of f (which is $\mathbb{R}$ if n is odd and $\mathbb{R}^{\geq 0}$ if n is even). Differentiate both sides of the equation

$$[f(x)]^n = x$$

(without substituting $f(x) = \sqrt[n]{x}$ into it!) with respect to x to obtain an equation for $f'(x)$.

First,

$$\frac{d}{dx}x = 1.$$

Now,

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$

by the chain rule. Hence

$$1 = n[f(x)]^{n-1}f'(x).$$

(b) Now, substitute $f(x) = \sqrt[n]{x}$ into the equation you got in part (a), simplify, and solve the equation for $f'(x)$. Did you get the expected result?

$$1 = n[f(x)]^{n-1}f'(x) = n\sqrt[n]{x}^{n-1}f'(x) = nx^{\frac{n-1}{n}}f'(x).$$

Hence

$$f'(x) = \frac{1}{n}\frac{1}{x^{\frac{n-1}{n}}} = \frac{1}{n}\frac{1}{x^{1-\frac{1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

This is exactly the result we would expect by using the power rule.