MCS 119 Final Exam Solutions

May 20, 2023

1. (10 pts) Find the n-th derivative of
$$f(x) = \frac{1}{x}$$
by calculating the first few derivatives and observing the pattern that occurs.

$$\begin{aligned}
\frac{d}{dx}\frac{1}{x} &= \frac{d}{dx}x^{-1} \\
&= (-1)x^{-1-1} \\
&= (-1)x^{-2} \\
\frac{d^2}{dx^2}x^{-1} &= \frac{d}{dx}[(-1)x^{-2}] \\
&= (-1)\frac{d}{dx}x^{-2} \\
&= (-1)(-2)x^{-2-1} \\
&= (-1)(-2)x^{-3} \\
\frac{d^3}{dx^3}x^{-1} &= \frac{d}{dx}[(-1)(-2)x^{-3}] \\
&= (-1)(-2)\frac{d}{dx}x^{-3} \\
&= (-1)(-2)(-3)x^{-3-1} \\
&= (-1)(-2)(-3)x^{-4} \\
\frac{d^4}{dx^4}x^{-1} &= \frac{d}{dx}[(-1)(-2)(-3)x^{-4}] \\
&= (-1)(-2)(-3)\frac{d}{dx}x^{-4} \\
&= (-1)(-2)(-3)(-4)x^{-4-1} \\
&= (-1)(-2)(-3)(-4)x^{-5}
\end{aligned}$$

Now, the pattern is quite clear. The exponent on the x decreases by 1 every time we differentiate. So after differentiating n times, it will be $-1-n$ or $-n-1$. The coefficient also shows a simple pattern: it is $(-1)(-2)(-3)\cdots(-n)$. So
$$\frac{d^n}{dx^n}x^{-1} = (-1)(-2)(-3)\cdots(-n)x^{-n-1}.$$
If you want to express this is a more compact form, then you can write
$$(-1)(-2)(-3)\cdots(-n) = (-1)^n 1\cdot 2\cdot 3\cdots n = (-1)^n n!$$
Hence
$$\frac{d^n}{dx^n}x^{-1} = (-1)^n n! x^{-n-1}.$$

The exercise did not ask you to prove that your conjecture is correct. But here is how we could check that it is. First of all, we can confirm that it works right at the beginning, when

$n = 1$. Then the pattern predicts

$$\frac{d^n}{dx^n} x^{-1} = (-1)^n n! x^{-n-1} = (-1)^1 1! x^{-1-1} = x^{-2}.$$

This matches our calculation of $\frac{d}{dx} x^{-1}$ above.

Now, we check that the pattern persists. That is if it correctly predicts $\frac{d^n}{dx^n} x^{-1}$, then it also correctly predicts $\frac{d^{n+1}}{dx^{n+1}} x^{-1}$. Suppose

$$\frac{d^n}{dx^n} x^{-1} = (-1)^n n! x^{-n-1}$$

for some positive integer n. For example, we know that this is correct for $n = 1$. Then

$$\frac{d^{n+1}}{dx^{n+1}} x^{-1} = \frac{d}{dx} \left(\frac{d^n}{dx^n} x^{-1} \right).$$

We assumed $\frac{d^n}{dx^n} x^{-1} = (-1)^n n! x^{-n-1}$, so

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}} x^{-1} &= \frac{d}{dx} \big((-1)^n n! x^{-n-1} \big) \\
&= (-1)^n n! \frac{d}{dx} x^{-n-1} \\
&= (-1)^n n! (-n-1) x^{-n-1-1} \\
&= (-1)^n n! (-(n+1)) x^{-n-2} \\
&= -(-1)^n \underbrace{n!(n+1)}_{(n+1)!} x^{-n-2} \\
&= (-1)^{n+1} (n+1)! x^{-n-2}.
\end{aligned}$$

This matches what our conjecture predicts. We can now argue that our conjecture is correct for $n = 1$, and therefore it must be correct for $n = 2$. So it must be correct for $n = 3$. So it must be correct for $n = 4$. Etc. We can conclude that it must be correct for every positive integer n.

2. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is the line with equation

$$\frac{x_0}{a^2} x + \frac{y_0}{b^2} y = 1$$

By implicit differentiation,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) &= \frac{d}{dx}1\\
\frac{2x}{a^2}+\frac{2y\frac{dy}{dx}}{b^2} &= 0\\
\frac{2y\frac{dy}{dx}}{b^2} &= -\frac{2x}{a^2}\\
\frac{y}{b^2}\frac{dy}{dx} &= -\frac{x}{a^2}\\
\frac{dy}{dx} &= -\frac{b^2x}{a^2y}
\end{aligned}$$

So at (x_0, y_0),

$$\frac{dy}{dx} = -\frac{b^2x_0}{a^2y_0},$$

and the equation of the tangent line is

$$y = -\frac{b^2x_0}{a^2y_0}\,x + c.$$

Since the tangent line passes through the point (x_0, y_0), we can find c by substituting (x_0, y_0) into the equation of the line:

$$\begin{aligned}
y_0 &= -\frac{b^2x_0}{a^2y_0}\,x_0 + c\\
c &= y_0 + \frac{b^2x_0}{a^2y_0}\,x_0 = \frac{a^2y_0^2}{a^2y_0} + \frac{b^2x_0^2}{a^2y_0} = \frac{a^2y_0^2+b^2x_0^2}{a^2y_0}
\end{aligned}$$

We also know that the point (x_0, y_0) is on the ellipse, so it satisfies the equation of the ellipse:

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2} = 1 \implies b^2x_0^2 + a^2y_0^2 = a^2b^2.$$

Substituting this into the numerator of the expression for c gives

$$c = \frac{a^2b^2}{a^2y_0} = \frac{b^2}{y_0}.$$

So the equation of the tangent line is

$$y = -\frac{b^2x_0}{a^2y_0}\,x + \frac{b^2}{y_0}.$$

To make this more symmetric, we can multiply by $\frac{y_0}{b^2}$:

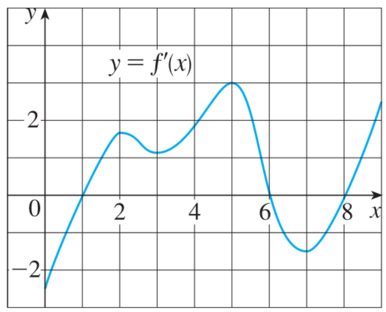
$$\frac{y_0}{b^2}\,y = -\frac{x_0}{a^2}\,x + 1.$$

Finally, we can add $\frac{x_0}{a^2}\,x$ to both sides:

$$\frac{y_0}{b^2}\,y + \frac{x_0}{a^2}\,x = 1$$

which is what we wanted to show.

3. (10 pts) The graph of the derivative f' of a continuous function f is shown below.



(a) (4 pts) At what values of x does f have a local maximum or minimum? How can you tell?

We know by Fermat's Theorem that if $x = c$ is a local extremum of f, then it must be a critical point of f. So either $f'(c)$ does not exist or $f'(c) = 0$. The graph of f' is continuous at every x for the interval $[0, 9]$ on which the graph is shown. So there is no place where $f'(x)$ does not exist. As for $f'(x) = 0$, that happens at $x = 1, 6, 8$. These critical points are the only candidates for the local extrema on $[1, 9]$. We can check if they actually are local extrema using the First Derivative Test.
At $x = 1$ and $x = 8$, the derivative f' changes from negative to positive. So f changes from decreasing to increasing. Therefore f has local minima at $x = 1$ and at $x = 8$. At $x = 6$, f' changes from positive to negative. So f changes from increasing to decreasing. Therefore f has a local maximum at $x = 6$.

(b) (4 pts) On what intervals is f concave upward or downward? How can you tell?

We know that f is concave up/down on an interval if f'' is positive/negative on that interval. We also know that f' is increasing/decreasing on an interval if its derivative f'' is positive/negative on that interval. So to tell where f is concave up, we are looking for intervals where f' is increasing. It is clear from the graph that f' is increasing on $(0, 2)$, $(3, 5)$, and $(7, 9)$. Similarly, f is concave down where f' is decreasing. From the graph, f' is decreasing on $(1, 3)$, and $(5, 7)$.
So f is concave up on $(0, 2)$, $(3, 5)$, and $(7, 9)$ and concave down on $(1, 3)$, and $(5, 7)$.

(c) (2 pts) State the x-coordinate(s) of the point(s) of inflection. How can you tell where the inflection point(s) are?

An inflection point is where f changes concavity from concave down to concave up or vice versa. Based on our asnwer to part (b), such points are at $x = 2, 3, 5, 7$.

4. (10 pts)
Use the definition of the derivative to find the derivative of $f(x) = \cos(x)$. Be sure to explain every step of your calculation.

Hint: The identity $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ may be useful.

$$\begin{aligned}\frac{d}{dx}\cos(x) &= \lim_{h\to 0}\frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h\to 0}\frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}\end{aligned}$$

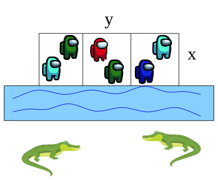
$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \cos(x) + \sin(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \left[\cos(x)\,\frac{\cos(h)-1}{h} - \sin(x)\,\frac{\sin(h)}{h}\right] \\
&= \lim_{h\to 0} \left[\cos(x)\,\frac{\cos(h)-1}{h}\right] - \lim_{h\to 0}\left[\sin(x)\,\frac{\sin(h)}{h}\right] && \text{by LL2} \\
&= \cos(x)\,\lim_{h\to 0}\frac{\cos(h)-1}{h} - \sin(x)\,\lim_{h\to 0}\frac{\sin(h)}{h} && \text{by LL3}
\end{aligned}
$$

Since we know from Section 1.4 that

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0,$$

we can now conclude

$$\frac{d}{dx}\cos(x) = \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x).$$



5. (10 pts) We would like to build a containment facility for amogi. We have amogus-proof fencing and a property with a crocodile-infested river. We can use the river as one side of our containment facility. It is easy to lose a foot or an arm in croc-infested waters, so amogi do not go in there. We will use the fence to build the other sides of a rectangular holding pen divided into three equal size compartments (see diagram), so we can sort amogi into three groups: those are not sus, those that are a little sus, and those that are really sus. We want the total area to be 10,000 ft^2. We want to use the least amount of fencing we can get away with. Find the dimensions of the optimal holding pen. Make sure you explain how you know that what you found is the absolute minimum of fencing required.

Based on the diagram to the left, the length of the fencing we need is

$$P = y + 4x.$$

The area of the fenced rectangle is

$$A = xy = 10000.$$

We can use this last equation to express y in terms of x:

$$10000 = xy \implies y = \frac{10000}{x}.$$

We can substitute this into P:

$$P = \frac{10000}{x} + 4x.$$

This is now a function of only one variable. We can find its minimum the usual way. First, note that $0 < x$ because x is one side of a rectangle, so it cannot be negative, and $x \neq 0$, otherwise xy could not equal 10000. In principle, there is no upper bound on x. So we are looking for the absolute minimum of P on the open interval $x \in (0, \infty)$. Let us see if we can find any local minimum of P. The derivative

$$P'(x) = \frac{d}{dx}(10000x^{-1} + 4x) = -10000x^{-2} + 4.$$

appears to exist for all $x \in (0, \infty)$. This makes sense as P is the sum of a rational function and a linear function with no discontinuities anywhere in $(0, \infty)$. So the only kind of critical point P may have on $(0, \infty)$ is where $P'(x) = 0$:

$$\begin{aligned} 0 &= -10000x^{-2} + 4 \\ 10000x^{-2} &= 4 \\ 10000 &= 4x^2 && \text{we know } x \neq 0 \\ 2500 &= x^2 \\ x &= 50 \end{aligned}$$

We need to check if this critical point is indeed the absolute minimum we are looking for. We will show that $P'(x)$ is negative for $0 < x < 50$ and $P'(x)$ is positive for $50 < x$. First, note that

$$P'(x) = -\frac{10000}{x^2} + 4 = \frac{4x^2 - 10000}{x^2}.$$

Since $x^2 > 0$ for all $x \in (0, \infty)$, the sign of $P'(x)$ is the same as the sign of the numerator.

If $0 < x < 50$, then

$$x^2 < 2500 \implies 4x^2 < 10000 \implies 4x^2 - 10000 < 0 \implies P'(x) < 0.$$

So P is decreasing on the interval $(0, 50)$. Similarly, if $x > 50$, then

$$x^2 > 2500 \implies 4x^2 > 10000 \implies 4x^2 - 10000 > 0 \implies P'(x) > 0.$$

So P is increasing on $(50, \infty)$. Hence $x = 50$ is both a local and an absolute minimum of P on the interval $(0, \infty)$.

If $x = 50$, then $y = 10000/x = 200$. So the optimal holding pen has dimensions $x = 50$ ft and $y = 200$ ft.

6. (5 pts each)

(a) Express the definite integral

$$\int_2^5 \frac{1}{x^3}\, dx$$

as the limit of a right-hand sum with a uniform partition. You do not need to evaluate the limit.

We will divide the interval $[2, 5]$ into n uniform subintervals. So each subinterval has width $\Delta x = \frac{5-2}{n} = \frac{3}{n}$. The i-th subinterval is

$$[x_{i-1}, x_i] = \left[2 + (i-1)\frac{3}{n}, 2 + i\frac{3}{n}\right] = \left[\frac{2n + 3(i-1)}{n}, \frac{2n+3i}{n}\right].$$

Since we want to set up a right-hand sum, the i-th sample point x_i^* is the right endpoint of the i-th subinterval

$$x_i^* = \frac{2n+3i}{n}.$$

Hence

$$\int_2^5 \frac{1}{x^3}\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\left(\frac{2n+3i}{n}\right)^3} \frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^3}{(2n+3i)^3} \frac{3}{n}.$$

(b) Use the Fundamental Theorem of Calculus to evaluate

$$\int_2^5 \frac{1}{x^3}\, dx$$

Since the antiderivative of $1/x^3$ is

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c,$$

the definite integral is

$$\int_2^5 \frac{1}{x^3}\,dx = -\frac{1}{2x^2}\bigg|_2^5 = -\frac{1}{50} + \frac{1}{8} = \frac{25-4}{200} = \frac{21}{200}.$$

7. **Extra credit problem.** In this exercise, you will use the limit of a Riemann sum to find

$$\int_1^2 \frac{1}{x^2}\,dx$$

by expressing it as the limit of a Riemann sum.

(a) (5 pts) First, express the Riemann sum using a uniform partitition with n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

and choose the sample points x_i^* to the geometric mean of the endpoints of its subinterval $[x_{i-1}, x_i]$. The geometric mean of two nonnegative numbers a and b is defined to be $\sqrt{ab}$.

We will divide the interval $[1, 2]$ into n uniform subintervals. So each subinterval has width $\Delta x = \frac{2-1}{n} = \frac{1}{n}$. The i-th subinterval is

$$[x_{i-1}, x_i] = \left[1 + (i-1)\frac{1}{n}, 1 + i\frac{1}{n}\right] = \left[\frac{n+(i-1)}{n}, \frac{n+i}{n}\right] = \left[\frac{n+i-1}{n}, \frac{n+i}{n}\right].$$

The i-th sample point is then

$$x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\frac{n+i-1}{n}\frac{n+i}{n}} = \sqrt{\frac{(n+i-1)(n+1)}{n^2}} = \frac{\sqrt{(n+i-1)(n+i)}}{n}$$

since we know n is positive. Hence the Riemann sum is

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \frac{1}{\left(\frac{\sqrt{(n+i-1)(n+i)}}{n}\right)^2}\frac{1}{n} = \sum_{i=1}^{n} \frac{n^2}{(n+i-1)(n+i)}\frac{1}{n} = \sum_{i=1}^{n} \frac{n}{(n+i-1)(n+i)}.$$

(b) (10 pts) Simplify the Riemann sum you got in part (a) by using the identity

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \qquad \text{if } k \neq 0, 1.$$

Then take the limit as $n \to \infty$ and evaluate the limit. Did you get the result you would expect?

Notice that

$$\frac{n}{(n+i-1)(n+i)} = n\frac{1}{(n+i-1)(n+i)} = n\left(\frac{1}{n+i-1} - \frac{1}{n+i}\right).$$

Hence

$$\begin{aligned}
\sum_{i=1}^{n} \frac{n}{(n+i-1)(n+i)} &= \sum_{i=1}^{n} n\left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \\
&= n\sum_{i=1}^{n}\left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \\
&= n\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3}\right. \\
&\qquad \left. + \cdots + \frac{1}{n+n-1} - \frac{1}{n+n}\right)
\end{aligned}$$

That is a telescoping sum, which simplifies to

$$\sum_{i=1}^{n} \frac{n}{i(i+1)} = n\left(\frac{1}{n} - \frac{1}{2n}\right) = n\,\frac{2-1}{2n} = n\,\frac{1}{2n} = \frac{1}{2}.$$

Hence

$$\int_1^2 \frac{1}{x^2}\,dx = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

since the $1/2$ is a constant and does not depend on n at all.

This matches what we would get by using the FTC:

$$\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx = \left.\frac{x^{-1}}{-1}\right|_1^2 = -\frac{1}{2} + \frac{1}{1} = \frac{1}{2}.$$