# Notes for Section 4.5

When we looked at antiderivatives and indefinite integrals in Sections 3.7 and 4.4, we saw a few examples we could solve by working backwards through the Chain Rule. This is fine in simple cases, but not so easy to do in more complicated cases. One way we can make it easier is by introducing notation that helps us remember how to work the Chain Rule backwards. This is the idea behind integration by substitution.

Suppose $F(x)$ is an antiderivative of the function $f(x)$. That is $F'(x) = f(x)$. Then

$$\int f(x)\,dx = F(x) + c$$

where $c$ is the usual arbitrary constant.

Now, let us try finding the antiderivative of the more complicated function $f(u(x))u'(x)$. It is easy enough to recognize this as the derivative of $F(u(x))$, but the point is to develop clever notation so we do not have to do such guesswork. Using $\frac{du}{dx}$ notation, $u'(x)$ is the same thing as $\frac{du}{dx}$. So

$$\int f(u(x))u'(x)\,dx = \int f(u(x))\,\frac{du}{dx}\,dx.$$

It is tempting to simplify $\frac{du}{dx}\,dx$ as $du$ and get

$$\int f(u(x))u'(x)\,dx = \int f(u(x))\,du.$$

Of course, that is not a valid thing to do because $\frac{du}{dx}$ is not actually a fraction, but a derivative. But let us see if it gives a result that is legitimate. Treating $u$ as a variable for now, note that

$$\int f(u)\,du = F(u) + c$$

is a to valid to say, since $F$ is indeed an antiderivate of $f$, and it does not matter if the input variable if called $x$ or $u$ or anything else. Now, recall that $u$ is also the output value of the function $u(x)$ for the input $x$. So the indefinite integral on the left-hand side of the last equation is in fact

$$\int f(u)\,du = \int f(u(x))\,du.$$

We have seen this before! It is just $\int f(u(x))u'(x)\,dx$. Let us put the pieces together:

$$\int f(u(x))u'(x)\,dx = \int f(u(x))\,du = \int f(u)\,du = F(u) + c.$$

We are not saying anything really new here. We already knew that

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$$

by the Chain Rule. So it makes good sense that $F(u)$ should be one of the antiderivatives of $f(u(x))u'(x)$, and hence the indefinite integral of $f(u(x))u'(x)$ is $F(u) + c$. But this does give us a streamlined way of doing the Chain Rule backwards. The idea is that if you need to find the indefinite integral of a function that looks like it might be the derivative of a composite function, then identify the inside function, and call it $u(x)$ (or any other letter you like). Then write

$$\frac{du}{dx} = u'(x) \implies du = u'(x)\,dx$$

and replace the part of the integrand that looks like $u'(x)\,dx$ with the much simpler $du$. At the same time, you would substitute $u$ for the inside function in the integrand.

Let us see an example. Let us find

$$\int 3x^2\sqrt{x^3+5}\,dx.$$

For sure $\sqrt{x^3+5}$ is a composite function, and the $3x^2$ factor does bear an uncanny resemblance to the derivative of the inside function. So we will set $u(x) = x^3+5$. Then

$$\frac{du}{dx} = 3x^2 \implies du = 3x^2\,dx.$$

So

$$\int 3x^2\sqrt{x^3+5}\,dx = \int \sqrt{\underbrace{x^3+5}_{u}}\,\underbrace{3x^2\,dx}_{du} = \int \sqrt{u}\,du.$$

That is an easy integral to do using the standard rule

$$\frac{d}{du}u^n = nu^{n-1} \implies \int u^n\,du = \frac{u^{n+1}}{n+1}+c$$

as long as $u \neq -1$. So

$$\int 3x^2\sqrt{x^3+5}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2}+c = \frac{2}{3}(x^3+5)^{3/2}+c.$$

Notice that we substituted $x^3+5$ back for $u$ in the end. This is because the original integrand $3x^2\sqrt{x^3+5}$ was a function of $x$, so it makes good sense that we would like to see its antiderivative in terms of $x$, and not in terms of the function $u$ we introduced along the way.

Make your way through Examples 1–4, and after studying them, try your own hand at integrating the same functions following this procedure. It should not be hard to do, but takes a bit of practice.

You can do the same thing with definite integrals. Let us try

$$\int_2^6 3x^2\sqrt{x^3+5}\,dx.$$

We will again let $u(x) = x^3+5$. Then

$$\frac{du}{dx} = 3x^2 \implies du = 3x^2\,dx.$$

So

$$\begin{aligned}
\int_2^6 3x^2\sqrt{x^3+5}\,dx &= \int_{x=2}^{x=6} \sqrt{u}\,du \\
&= \left[\frac{2}{3}u^{3/2}\right]_{x=2}^{x=6} \\
&= \left[\frac{2}{3}(x^3+5)^{3/2}\right]_{x=2}^{x=6} \\
&= \frac{2}{3}\left((6^3+5)^{3/2}-(2^3+5)^{3/2}\right) \\
&= \frac{2}{3}\left(221^{3/2}-13^{3/2}\right).
\end{aligned}$$

Notice that I added an $x =$ in front of the lower and upper bounds after the substitution step. This is because it would be a mistake to substitute 2 and 6 for $u$ in the end. Remember that $u(x) = x^3+5$. So when $x = 2$, $u(x) = 2^3+5 = 13$, and when $x = 6$, $u(x) = 6^3+5 = 221$.

Another way we could have kept track of the appopriate bounds would have been to convert them to $u(2)$ and $u(6)$ along the way:

$$\int_2^6 3x^2\sqrt{x^3+5}\,dx = \int_{u(2)}^{u(6)} \sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_{13}^{221} = \frac{2}{3}\left(221^{3/2} - 13^{3/2}\right).$$

This is actually a little quicker and easier in most cases. The book even states this as a theorem and proves it. The proof is nothing more than doing substitution and the Chain Rule correctly. You can see another example of this approach in Example 5.

The section ends with a discussion of how the symmetry of functions and definite integrals are related. We have talked about this in the specific context of the sine function. The observations should be easy to follow.