MCS 119 Final Exam

All of your answers must be carefully justified. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. You may refer to any result proved in class unless otherwise specified. You may use results you proved on your homework, except for ones the problem specifically asks you to prove.

You are not allowed to use your textbook or your class notes, but you may use a simple calculator.

1. (a) (7 pts) Use the definition of the derivative to find the derivative of $f(x) = \sqrt{6-x}$. Be sure to explain every step of your calculation.

   (b) (3 pts) What are the domains of f and f'? Do not forget to explain why.

2. (10 pts) Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

3. (5 pts each)
   (a) Find an approximation to the integral

   $$\int_0^4 (x^2 - 3x)\,dx$$

   using a Riemann sum with right endpoints and a uniform partition with 8 subintervals.

   (b) Use the Evaluation Theorem (or the Fundamental Theorem of Calculus) to evaluate

   $$\int_0^4 (x^2 - 3x)\,dx.$$

4. Let f be a function of real numbers and a a real number.
   (a) (3 pts) Define what it means for f to be differentiable at a.

   (b) (7 pts) Prove that if f is differentiable at a then f must be continuous at a.

5. (10 pts) The Row vs. Wade problem. As part of a team bulding exercise, the nine justices of the Supreme Court of the United States find themselves on the bank of a long and straight canal that is a mile wide. Their task is to reach Point Consensus (point C in the diagram), which is on the opposite bank of the canal, 2 miles from the point straight across from where they are standing. There is a boat they can row across the canal from point A to some point B on the other side. The boat is old and has a major leak (ha ha, pun intended), and therefore they can row it at a speed of only 2 miles/h. The opposite bank of the canal is a steep cliff the justices cannot climb. But the canal is shallow there and they can wade along the bank from point B to point C at 2.5 miles/h. Find the optimal place of point B so the justices reach Point Consensus in the least amount of time by rowing across the canal and then wading along the bank. What is the minimal time for them to complete the trip? How do you know that what you found is the absolute minimum?

Point
Consensus

A' B C

wade

1 mile row

2 miles

A
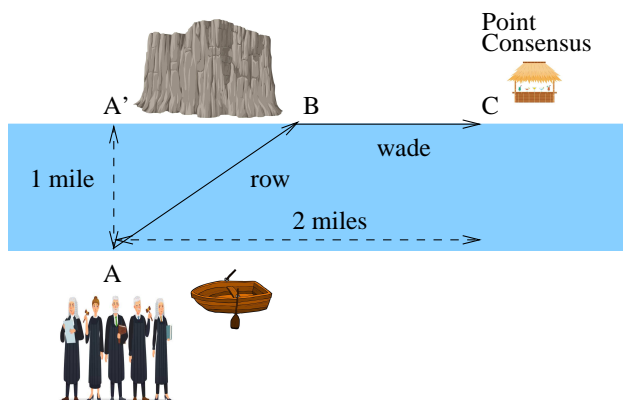
6. (10 pts) In his relentless quest to catch the Road Runner, Wile E. Coyote is testing his ACME® rocket. The rocket is equipped with an accelerometer, which registers acceleration of $a(t) = 8 + 3t$, measured in m/s^2, t seconds after its launch. The rocket launches from rest at time $t = 0$ and has enough fuel for 10 seconds. How far can the coyote fly in the 10 seconds before the rocket's fuel is exhausted?

7. **Extra credit problem.**

(a) (10 pts) Let f be a function or real numbers and $a \in \mathbb{R}$ such that
$$\lim_{x \to a} f(x) = \infty.$$
Prove that
$$\lim_{x \to a} \frac{1}{f(x)} = 0.$$
Hint: you will want to use the formal definition of the limit (as in $\delta - \epsilon$) and the infinite limit (as in $\delta - M$) for this argument.

(b) (5 pts) Is it also true that if
$$\lim_{x \to a} f(x) = 0$$
then
$$\lim_{x \to a} \frac{1}{f(x)} = \infty?$$
If you think it is true, prove it; if you do not think it it true, find a counterexample.