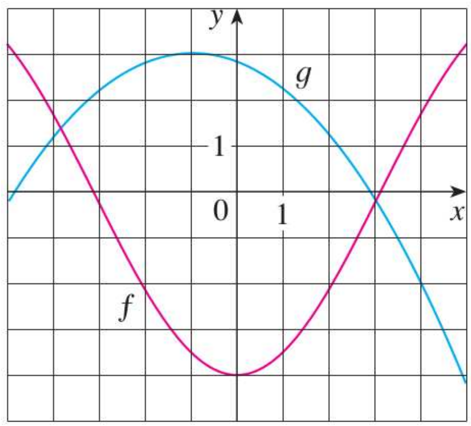
MCS 121 Exam 1 Solutions

Oct 12, 2017

1. (10 pts) Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \ldots, 5$.

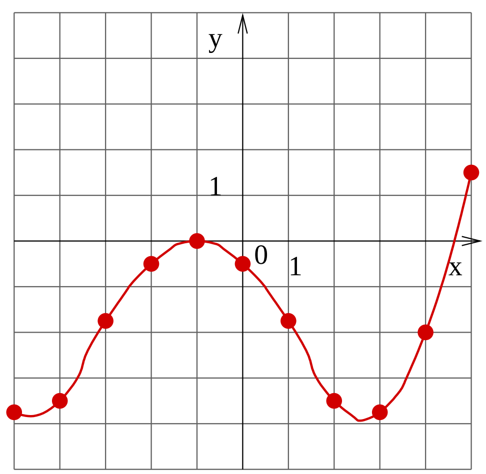


Use these estimates to sketch a rough graph of $f \circ g$ below.

First, let's make a table of values by reading them off the graphs. Note that these values are approximate.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
g(x)	-0.2	1	2.2	2.8	3	2.8	2.2	1	-0.2	-2	-4
f(g(x))	-3.9	-3.5	-1.9	-0.5	0	-0.5	-1.9	-3.5	-3.9	-2	1.5

Now we just need to plot the points $\big(x, f(g(x))\big)$ in the xy-plane and connect them with a curve.



2. (10 pts) When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a}).$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)

Find the inverse of this function and explain its meaning.

Let's just write Q instead of $Q(t)$ as we find the inverse by solving the equation $Q = Q_0(1 - e^{-t/a})$ for t. So

$$\begin{aligned}
Q &= Q_0(1 - e^{-t/a}) \\
\frac{Q}{Q_0} &= 1 - e^{-t/a} \\
1 - \frac{Q}{Q_0} &= e^{-t/a} \\
\ln\left(1 - \frac{Q}{Q_0}\right) &= -\frac{t}{a} \\
-a\ln\left(1 - \frac{Q}{Q_0}\right) &= t
\end{aligned}$$

So the inverse function is

$$t(Q) = -a\ln\left(1 - \frac{Q}{Q_0}\right).$$

The inverse function gives the time (in seconds) it takes to charge the flash to a charge of Q.

Note that whatever notation you use to solve this problem, in the end the t and the Q must switch roles. So for the inverse function, Q is the independent variable and t is the dependent variable.

3. (5 pts each)

(a) Give an example of a function. Make sure you specify the domain and the codomain and carefully describe what the function does to its input. Explain why your example satisfies the definition of function.

Here is an example from class. Let S be the set of all students at GAC. Let $T = \mathbb{Z}$, the set integers. Let $f : S \to T$ be

$$f(x) = \text{x's GAC student ID number.}$$

Then f is a function because it assigns to every student in S one and only one number in T.

(b) Give an example of a rule that is not a function. Just like in part (a), make sure your description of the rule is carefully explained and that you fully justify why the rule fails to satisfy the definition of function.

Let S be the set of all students at GAC. Let T be the set of all pets. Let

$$f(x) = \text{x's pet.}$$

This is not a function because some students have no pets at all while others have more than one. So $f(x)$ has no value for some x and has more than one value for others, while a function must have exactly one value for every input.

4. (5 pts each) As Homer Simpson winds down with a few pints of Duff at Moe's Tavern after a long week at work, his IQ is given by

$$q(x) = 64 \cdot (3/4)^x$$

where x is the amount of beer Homer has consumed measured in quarts.

(a) Find the average rate of change of Homer's IQ as he chugs another pint after already downing six pints. (One quart is two pints).

6 pints are 3 quarts, so when Homer has had six pints, $x_1 = 3$. After another pint, $x_2 = 3.5$. The average rate of change of q is

$$\begin{aligned}
\frac{\Delta q}{\Delta x} &= \frac{q(x_2) - q(x_1)}{x_2 - x_1} \\
&= \frac{64\,\left(\frac{3}{4}\right)^{3.5} - 64\,\left(\frac{3}{4}\right)^{3}}{3.5 - 3} \\
&= \frac{64\,\left(\frac{3}{4}\right)^{3}\left(\left(\frac{3}{4}\right)^{1/2} - 1\right)}{1/2} \\
&= 2\left(64\,\frac{27}{64}\left(\sqrt{\frac{3}{4}} - 1\right)\right) \\
&= 54\left(\frac{\sqrt{3}}{2} - 1\right) \\
&= 27(\sqrt{3} - 2) \\
&\approx -7.23
\end{aligned}$$

So the average rate of change of Homer's IQ from beer #6 to beer #7 is about -7.23/qt. Note that the number by itself without physical units does not make sense as an answer to the question. Also, the answer to a word problem should be a complete sentence.

(b) Explain how you could estimate the instantaneous rate of change of Homer's IQ at right after finishing pint #6. You do not need to find the rate, only describe how you would do it.

We would take two points $x_1 = 3$ and $x_2 = 3 + \Delta x$ where Δx is some small number. Then the difference quotient

$$\frac{\Delta q}{\Delta x} = \frac{q(x_2) - q(x_1)}{x_2 - x_1} = \frac{q(3 + \Delta x) - q(3)}{\Delta x}$$

gives the slope of the secant line between $x_1 = 3$ and $x_2 = 3 + \Delta x$. We could calculate this for a few values of Δx (both positive and negative) as $\Delta x \to 0$ and see what number the slopes are approaching. Once the slope does not change much as we make Δx closer to 0, we can use the slope of the secant line as an approximation to the slope of the tangent line.

If you are curious, the instantaneous rate of change at $x = 3$ is about -7.77/qt. We will learn this semester how to calculate it quickly, easily, and precisely.

5. (5 pts each) **Extra credit problem.** Let $f : A \to B$, $g : A \to B$ and $h : B \to C$.

(a) Find an example to show that if $h \circ f$ and $h \circ g$ are the same function, this does not imply f and g are the same function. That is we could have $h \circ f = h \circ g$ while $f \neq g$.

Let $A = B = C = \mathbb{R}$ and $f(x) = x$, $g(x) = -x$, and $h(x) = x^2$. Note that $f \neq g$, e.g. $f(2) = 2$ and $g(2) = -2$. Now $h \circ f$ and $h \circ g$ are both also functions $\mathbb{R} \to \mathbb{R}$ and

$$h \circ f(x) = h(f(x)) = x^2$$
$$h \circ g(x) = h(g(x)) = (-x)^2 = x^2$$

So we know $h(f(x)) = h(g(x))$ for every possible input $x \in \mathbb{R}$. Hence $h \circ f = h \circ g$.

(b) Prove that if h is a one-to-one function, then $h \circ f = h \circ g$ does in fact imply $f = g$.

Let h be a one-to-one function and f and g functions such that $h \circ f = h \circ g$. We need to show that $f = g$, that is $f(x) = g(x)$ for every $x \in A$. We know that for every $x \in A$, we have $h(f(x)) = h(g(x))$. Since h is one-to-one, $h(f(x)) = h(g(x))$ implies $f(x) = g(x)$. This is true for any input x. Therefore $f = g$.