MCS 121 Final Exam Solutions

Section 4, 1:30-2:20 MTuThF

Dec 19, 2017

1. Suppose g is an odd function and let $h = f \circ g$.

   (a) (4 pts) Is h always an odd function?

   Not always. E.g. if $f(x) = x^2$ and $g(x) = x^3$ then
   $$h(x) = f \circ g(x) = \left(x^3\right)^2 = x^6,$$
   which we know is not an odd function since $h(-x) = (-x)^6 = x^6 \neq -x^6 = -h(x)$ for any nonzero x.
   Notice that this one example is enough to show that h is **not always** am odd function.

   (b) (3 pts) What if f is odd? Is h odd, even, or neither in that case?

   If f is also odd, then h will always be odd. We cannot prove this with just one example because there are infinitely many ways to choose the odd functions f and g. We need a general argument. Here is one. If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all x. Hence
   $$h(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -h(x)$$
   for all x. This shows that h is an odd function.

   (c) (3 pts) What if f is even? Does that make h odd, even, or neither?

   In this case, h will always be an even function. If f is even and g is odd, then $f(-x) = f(x)$ and $g(-x) = -g(x)$ for all x. Hence
   $$h(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = h(x)$$
   for all x. This shows that h is an even function.

2. (5 pts each)

   (a) Use the Limit Laws to find the value of
   $$\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x}.$$

   First, it is tempting to use Limit Law #5 to rewrite the limit of this quotient as the quotient of two limits. But that does not work because the limit in the denominator $\lim_{x\to 0} x = 0$ and LL #5 cannot be used in such a case. So the first thing we will do is multiply both the numerator and the denominator by $\sqrt{3+x} + \sqrt{3}$ to rationalize the numerator:
   $$\begin{aligned}
   \frac{\sqrt{3+x} - \sqrt{3}}{x} &= \frac{\sqrt{3+x} - \sqrt{3}}{x} \, \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} \\
   &= \frac{\sqrt{3+x}^2 - \sqrt{3}^2}{x(\sqrt{3+x} + \sqrt{3})} \\
   &= \frac{3 + x - 3}{x(\sqrt{3+x} + \sqrt{3})} \\
   &= \frac{x}{x(\sqrt{3+x} + \sqrt{3})}
   \end{aligned}$$

$$= \frac{1}{\sqrt{3+x}+\sqrt{3}}$$

where we can cancel the x because $x \neq 0$ for our purposes, as we are interested in what $\frac{\sqrt{3+x}-\sqrt{3}}{x}$ approaches as x gets close to 0 but not in what happens when $x = 0$. Now

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x} &= \lim_{x\to 0} \frac{1}{\sqrt{3+x}+\sqrt{3}} \\
&= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}(\sqrt{3+x}+\sqrt{3})} && \text{LL \#5} \\
&= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}} && \text{LL \#1} \\
&= \frac{1}{\lim_{x\to 0}\sqrt{3+x}+\sqrt{3}} && \text{LL \#7} \\
&= \frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}} && \text{LL \#11} \\
&= \frac{1}{\sqrt{3+0}+\sqrt{3}} && \text{Direct Substitution Property} \\
&= \frac{1}{2\sqrt{3}}
\end{aligned}$$

(b) Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is less than the degree of Q.

We will show that

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = 0.$$

Rather than giving a general argument with indices and complex notation, I will give an argument through an example. Notice that the example is general enough that the same argument works in every case. Let $P(x) = 3x^2 + 5$ and $Q(x) = 2x^4 - 6x^3 + 4x$. Then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{3x^2+5}{2x^4-6x^3+4x}$$

Divide both the numerator and the denominator by x^4. We can do this because dividing the numerator and the denominator of a fraction by the same nonzero number does not change its value, and as $x \to \infty$, we may assume $x \neq 0$. So

$$\lim_{x\to\infty} \frac{3x^2+5}{2x^4-6x^3+4x} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^4}+\frac{5}{x^4}}{\frac{2x^4}{x^4}-\frac{6x^3}{x^4}+\frac{4x}{x^4}} = \lim_{x\to\infty} \frac{\frac{3}{x^2}+\frac{5}{x^4}}{2-\frac{6}{x}+\frac{4}{x^3}}.$$

Now, as $x \to \infty$, the powers x^1, x^2, x^3, x^4 all go toward ∞. This makes the terms $\frac{3}{x^2}, \frac{5}{x^4}, \frac{6}{x}, \frac{4}{x^3}$ approach 0 as the numerators are fixed numbers while the denominators get arbitrarily large. So using Limit Laws #1 and #2,

$$\lim_{x\to\infty} \left(\frac{3}{x^2}+\frac{5}{x^4}\right) = 0 + 0 = 0$$

and

$$\lim_{x\to\infty} \left(2-\frac{6}{x}+\frac{4}{x^3}\right) = 2 + 0 + 0 = 2$$

And now by Limit Law #5,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \frac{0}{2} = 0.$$

In any other case, you can do the same by dividing $P(x)$ and $Q(x)$ by the largest power of x in Q. Since the terms in P all have lower degree, this will make the numerator approach 0 while the denominator approaches a specific nonzero number.

(c) Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is more than the degree of Q.

The argument is similar to part (b). We will show that

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \pm\infty.$$

Notice that the example below is general enough that the same argument works in every case. Let $P(x) = 2x^4 - 6x^3 + 4x$ and $Q(x) = 3x^2 + 5$. Then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{2x^4 - 6x^3 + 4x}{3x^2 + 5}$$

Divide both the numerator and the denominator by x^2. We can do this because dividing the numerator and the denominator of a fraction by the same nonzero number does not change its value, and as $x \to \infty$, we may assume $x \neq 0$. So

$$\lim_{x\to\infty} \frac{2x^4 - 6x^3 + 4x}{3x^2 + 5} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} + \frac{5}{x^2}}{\frac{2x^4}{x^2} - \frac{6x^3}{x^2} + \frac{4x}{x^2}} = \lim_{x\to\infty} \frac{3 + \frac{5}{x^2}}{2x^2 - 6x + \frac{4}{x}}.$$

Now, as $x \to \infty$, the powers x^1 and x^2 all go toward ∞. This makes the terms $\frac{5}{x^2}, \frac{4}{x}$ approach 0 as the numerators are fixed numbers while the denominators get arbitrarily large. So using Limit Law #1

$$\lim_{x\to\infty} \left(3 + \frac{5}{x^2}\right) = 3 + 0 = 3.$$

As for the denominator, the $2x^2$ term goes to ∞ as $x \to \infty$. The $-6x$ term goes toward $-\infty$, but the $2x^2$ term grows faster and eventually dominates, so $2x^2 - 6x \to \infty$ as $x \to \infty$. So

$$\lim_{x\to\infty} \left(2x^2 - 6x + \frac{4}{x}\right) = \infty$$

while the numerator approaches 2. A large number divided by 2 is still a large number, so

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \infty$$

in this case.

In any other case, you can do the same by dividing $P(x)$ and $Q(x)$ by the largest power of x in Q. Since some terms in P have higher degree, this will make the numerator approach ∞, or $-\infty$ if the coefficient of the highest degree term is negative, while the denominator approaches a specific nonzero number. Hence the limit will always be ∞ or $-\infty$.

3. (10 pts) Where is the function $h(x) = |x-1| + |x+2|$ differentiable? Give a formula for $h'(x)$ and sketch the graphs of h and h'.

Notice that if $x \geq 1$ then $x+2 > 0$ and $x - 1 \geq 0$, so

$$|x-1| + |x+2| = x - 1 + x + 2 = 2x + 1.$$

Similarly, if $x < -2$ then $x + 2 < 0$ and $x - 1 < 0$, so

$$|x-1| + |x+2| = -(x-1) - (x+2) = -x + 1 - x - 2 = -2x - 1.$$

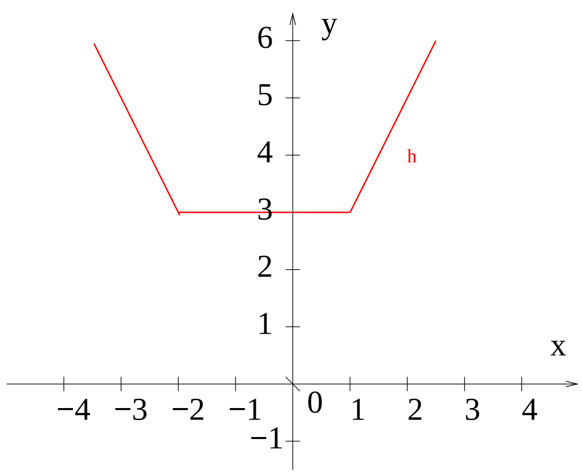
Finally, if $-2 \leq x < 1$ then $x - 1 < 0$ and $x + 2 \geq 0$, so

$$|x-1| + |x+2| = -(x-1) + x + 2 = -x + 1 + x + 2 = 3.$$

So h can also be expressed as the piecewise defined function

$$h(x) = \begin{cases} 2x+1 & \text{if } 1 \leq x \\ 3 & \text{if } -2 \leq x < 1 \\ -2x-1 & \text{if } x < -2 \end{cases}$$

Here is the graph of h:



It is easy to see either from the graph, or by differentiating h on the various pieces, that the slope of h is -2 at $x < -2$, 0 at $-2 < x < 1$, and 2 at $1 < x$. At $x = -2$ and at $x = 1$, the graph has a sharp corner, so h is not differentiable at these points. More precisely,

$$\lim_{x \to 1} \frac{h(x) - h(1)}{x - 1}$$

does not exist because the left and right limits are different:

$$\lim_{x \to 1^-} \frac{h(x) - h(1)}{x - 1} = \lim_{x \to 1^-} \frac{3 - 3}{x - 1} = \lim_{x \to 1^-} \frac{0}{x - 1} = 0,$$

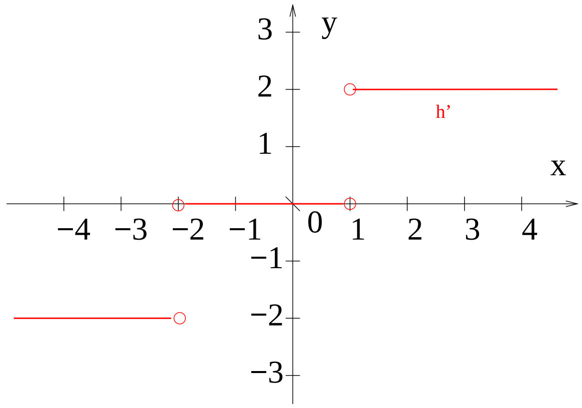
as $h(x) = 3$ when x is close to 1 but $x < 1$. But

$$\lim_{x \to 1^+} \frac{h(x) - h(1)}{x - 1} = \lim_{x \to 1^-} \frac{2x + 1 - 3}{x - 1} = \lim_{x \to 1^-} \frac{2x - 2}{x - 1} = \lim_{x \to 1^-} 2 = 2,$$

as $h(x) = 2x + 1$ when x is close to 1 but $x > 1$. Similarly, the left and right derivates at $x = -2$ are -2 and 0. So

$$h'(x) = \begin{cases} 2 & \text{if } 1 < x \\ 0 & \text{if } -2 < x < 1 \\ -2 & \text{if } x < -2 \end{cases}$$

Here is the graph of h':



Since $h'(x)$ exists for every real number x except $x = -2$ and $x = 1$, the function h is differentiable everywhere but $x = -2$ and $x = 1$.

4. (10 pts) Let f be a function (real inputs, real values). Prove that if f is differentiable at $x = a$ then f must be continuous at $x = a$.

This is Theorem 4 in Section 2.8 in your textbook. Here is the argument we gave in class. Let f be a function that is differentiable at $x = a$. We will first show that

$$\lim_{x \to a} [f(x) - f(a)] = 0.$$

We do this by multiplying $f(x) - f(a)$ by $\frac{x-a}{x-a}$, which we can do because $\frac{x-a}{x-a} = 1$ whenever $x - a \neq 0$. Inside the limit, we are interested in what happens as x gets close to a but not in what happens at $x = a$, so we may assume $x - a \neq 0$.

$$\begin{aligned}
\lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \left[(f(x) - f(a)) \frac{x-a}{x-a} \right] \\
&= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x-a} (x-a) \right] \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \lim_{x \to a} (x-a) && \text{LL \#4} \\
&= f'(a) \cdot 0 = 0
\end{aligned}$$

Notice we could use LL #4 above because we knew

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a)$$

exists because , and we knew $\lim_{x \to a}(x - a) = 0$ by the Direct Substitution Property of polynomials. Now, by LL #1,

$$\lim_{x \to a} f(x) = \lim_{x \to a} [f(x) - f(a) + f(a)] = \lim_{x \to a} [f(x) - f(a)] + \lim_{x \to a} f(a) = 0 + f(a) = f(a).$$

This show f is continuous at $x = a$.

5. (a) (8 pts) Let $f(x) = 1/x$. Use the definition of the derivative to find $f'(x)$. Then use your result to find the equation of the tangent line to f at $x = 3$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2},
\end{aligned}
$$

where the last step is by the Direct Substitution Property of rational functions. Therefore the slope of the tangent line to f at $x = 3$ is $f'(3) = -1/3^2 = -1/9$. Hence the equation of the tangent line is $y = -x/9 + b$. The tangent line passes through $(3, 1/3)$, so

$$
\frac{1}{3} = -\frac{3}{9} + b \implies \frac{1}{3} + \frac{3}{9} = b \implies b = \frac{2}{3}.
$$

So the equation of the tangent line is $y = -x/9 + 2/3$.

(b) (7 pts) Find the derivative of

$$
f(x) = \frac{\ln(x)}{e^{x^2}}.
$$

You can do this with the quotient rule and the chain rule, or the product rule and the chain rule. I will do the latter:

$$
\begin{aligned}
\frac{d}{dx}\frac{\ln(x)}{e^{x^2}} &= \frac{d}{dx}\big(\ln(x)e^{-x^2}\big) & \\
&= \frac{d}{dx}\ln(x)e^{-x^2} + \ln(x)\frac{d}{dx}e^{-x^2} & \text{product rule} \\
&= \frac{1}{x}e^{-x^2} + \ln(x)e^{-x^2}(-2x) & \text{chain rule} \\
&= \frac{\frac{1}{x} - 2x\ln(x)}{e^{x^2}}. &
\end{aligned}
$$

6. (10 pts) Looking at his list of who has been naughty and who has been nice, Santa says "how interesting, if I add the square of the percentage of children who have been naughty to the square of the percentage of children who have been nice, the result I get is the smallest it could possibly be." What are the two percentages?

Notice that this is the same problem we did in class when we found the smallest possible number that is the sum of the squares of two nonnegative numbers whose sum is 16 (problem 4.7.4 in your textbook), only with different numbers and a Santa story. So, let x and y be the percentages of children who have been naughty and who have been nice respectively. We are looking for values of x and y such that $x^2 + y^2$ is minimal. Since $x + y = 100\% = 1$, we can solve for $y = 1 - x$ and subsitute it into

$$
x^2 + y^2 = x^2 + (1-x)^2 = x^2 + 1 - 2x + x^2 = 2x^2 - 2x + 1
$$

Let $f(x) = 2x^2 - 2x + 1$. We want to find the global minimum of f on the interval $x \in [0, 1]$. By Fermat's Theorem, the global minimum must be at an endpoint of the closed interval or at a critical point of f. Since f is a polynomial, it is differentiable at every real number, so the only kind of critical point it has is where its derivative $f'(x) = 2(2x) - 2 + 0 = 4x - 2$ is 0. So

$$4x - 2 = 0 \implies 4x = 2 \implies x = \frac{1}{2}.$$

The values of f at this critical point and at $x = 0$ and $x = 1$ are

$$\begin{aligned} f(0) &= 2(0^2) - 2(0) + 1 = 1 \\ f(1/2) &= 2(1/2)^2 - 2(1/2) + 1 = 1/2 \\ f(1) &= 2(1^2) - 2(1) + 1 = 1. \end{aligned}$$

That is the miminum is at $x = 1/2$. This gives $y = 1/2$, so the percentages of naughty and nice children on Sahta's list were both 50%.

7. **Extra credit problem.** The purpose of this exercise is to prove the Squeeze Theorem: if f, g, and h are functions such that

$$f(x) \leq g(x) \leq h(x)$$

for every x in some neighborhood of b, and there is a real number L such that

$$\lim_{x\to b} f(x) = \lim_{x\to b} h(x) = L,$$

then

$$\lim_{x\to b} g(x) = L.$$

So let f, g, and h be functions and L a real number that satisfy the two conditions above.

(a) (10 pts) First, notice that $\lim_{x\to b}[h(x) - f(x)] = 0$ by Limit Law #2. Use this to show that given any $\epsilon > 0$, there is a corresponding $\delta > 0$ such that if $0 < |x - b| < \delta$ then $|g(x) - f(x)| < \epsilon$. Conclude from this that $\lim_{x\to b}[g(x) - f(x)] = 0$.

Let ϵ be any positive real number. Since $\lim_{x\to b}[h(x) - f(x)] = 0$, we know that for every such ϵ, there exists a $\delta > 0$ such that if $0 < |x - b| < \delta$ then $|h(x) - f(x)| < \epsilon$. We also know

$$f(x) \leq g(x) \leq h(x)$$

for every x near b, and subtracting $f(x)$ from all sides gives

$$0 \leq g(x) - f(x) \leq h(x) - f(x).$$

So $h(x) - f(x)$ is actually nonnegative and therefore $|h(x) - f(x)| = h(x) - f(x)$. We know this is smaller than ϵ. We now have

$$0 \leq g(x) - f(x) \leq h(x) - f(x) < \epsilon.$$

Therefore

$$|g(x) - f(x)| = g(x) - f(x) < \epsilon.$$

Since we can make $|g(x) - f(x)| < \epsilon$ for any $\epsilon > 0$ by choosing an appropriate δ and requiring that $0 < |x - b| < \delta$, we know that $\lim_{x\to b}[g(x) - f(x)] = 0$.

(b) (5 pts) Use your result from part (a) to prove that $\lim_{x\to b} g(x) = \lim_{x\to b} f(x) = L$. If you did not know how to do part (a), you may still use $\lim_{x\to b}[g(x) - f(x)] = 0$ to do part (b).

Notice that

$$\begin{aligned}
\lim_{x\to b} g(x) &= \lim_{x\to b}\,[g(x) - f(x) + f(x)] \\
&= \lim_{x\to b}\,[g(x) - f(x)] + \lim_{x\to b} f(x) && \text{LL \#1} \\
&= 0 + L = L
\end{aligned}$$