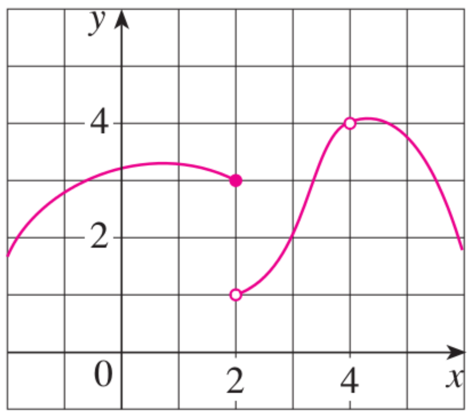
MCS 121 EXAM 2 SOLUTIONS

Nov 9, 2017

1. (10 pts) Use the given graph of $f$ to state the value of each quantity, if it exists. If the limit exists, explain why it is what you say it is; if it does not exist, explain why it does not exist.

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$

(d) $f(2)$ (e) $\lim_{x\to 4} f(x)$ (f) $f(4)$




(a) $\lim_{x\to 2^-} f(x) = 3$ because as $x$ approaches 2 from the left, the values of $f$ get arbitrarily close to 3. Note that the fact that the values actually reach 3 as $x = 2$ is irrelevant.
(b) $\lim_{x\to 2^+} f(x) = 1$ because as $x$ approaches 2 from the right, the values of $f$ get arbitrarily close to 1.
(c) $\lim_{x\to 2} f(x)$ does not exist because the left and right limits of $f$ as $x$ approaches 2 are different.
(d) $f(2) = 3$ because the filled circle indicates that $(2, 3)$ is a point on the graph of $f$.
(e) $\lim_{x\to 4} f(x) = 4$ because as $x$ approaches 4, the values of $f$ get arbitrarily close to 4.
(f) $f(4)$ does not exist because there is no point on the graph whose $x$ coordinate is 4.

2. (5 pts each)

(a) What is wrong with the following equation?

$$\frac{x^2+x-6}{x-2} = x+3$$

If $x = 2$, the left-hand side is $0/0$, which is undefined, while the right-hand side is 5. So the two sides are not equal.

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} (x+3)$$

is correct.

This is correct because the limit is taken as $x$ gets close to 2. As far as the limit goes, the value of the function when $x = 2$ is irrelevant. And at any $x \neq 2$, it is indeed true that

$$\frac{x^2+x-6}{x-2} = x+3.$$

Note that this is why in the $\delta - \epsilon$ definition of the limit, we have $0 < |x-b| < \delta$ and not only $|x-b| < \delta$.

3. (a) (4 pts) Let $f$ be a function and $b$ a real number. Define what

$$\lim_{x \to b} f(x) = \infty$$

means.

$\lim_{x\to b} f(x) = \infty$ means that the values of $f$ get arbitrarily large as $x$ gets sufficiently close to $b$. More precisely, for any real number $M$, however large, there is a $\delta > 0$ such that if $0 < |x-b| < \delta$ then $f(x) > M$.

(b) (6 pts) Give an example of a function $f$ and a real number $b$ such that

$$\lim_{x \to b} f(x) = \infty.$$

Make sure you carefully explain why your example does indeed have a limit of infinity as $x$ approaches $b$.

For example, $\lim_{x\to 0} \frac{1}{x^2} = \infty$. When $x$ is a number that is close to 0, $x^2$ will be even closer to 0 and it will be positive. So $1/x^2$ will be a large positive number. In fact, we can make $1/x^2$ as big as we want by making $x$ get sufficiently close to 0.
Here is a more precise argument. Let $M$ be any positive real number. Since we want to show that $1/x^2$ can be arbitrarily large, negative values of $M$ are not really relevant. So we might as well assume that $M > 0$. We want to find how close $x$ has to be to 0 so $1/x^2 > M$. By solving this inequality, we get

$$\begin{aligned} \frac{1}{x^2} &> M \\ \frac{1}{M} &> x^2 \\ \sqrt{\frac{1}{M}} &> \sqrt{x^2} \\ \frac{1}{\sqrt{M}} &> |x| \end{aligned}$$

So if $0 < |x| < 1/\sqrt{M}$, that is $x$ is closer to 0 than a distance of $1/\sqrt{M}$ but $x$ is not 0, then $1/x^2 > M$. No matter how large $M$ is, $1/\sqrt{M}$ will be some positive distance.

4. (10 pts) Let $f(x) = \sqrt{x}$. Use the limit laws to evaluate

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Justify each step of your work by referring to the limit law(s) you are using.

First, we substitute $x$ and $x+h$ into $f$,:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

We will now multiply $\frac{\sqrt{x+h}-\sqrt{x}}{h}$ by $\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$, which we can do because $\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = 1$ unless the numerator and the denominator are both 0. But that would only happen if $x = h = 0$.

In fact, we are not interested in what happens when $h = 0$ because the limit is about what happens to the values of $\frac{f(x+h)-f(x)}{h}$ as $h$ gets close to 0, but not at $h = 0$. So

$$\begin{aligned}
&\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&\qquad = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&\qquad = \lim_{h\to 0} \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&\qquad = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&\qquad = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} && \text{we can cancel } h \text{ since } h \neq 0 \\
&\qquad = \frac{\lim_{h\to 0} 1}{\lim_{h\to 0}\left(\sqrt{x+h}+\sqrt{x}\right)} && \text{by Limit Law \#5} \\
&\qquad = \frac{1}{\lim_{h\to 0}\sqrt{x+h}+\lim_{h\to 0}\sqrt{x}} && \text{by Limit Laws \#7 and \#2} \\
&\qquad = \frac{1}{\sqrt{\lim_{h\to 0}(x+h)}+\sqrt{\lim_{h\to 0} x}} && \text{by Limit Law \#11} \\
&\qquad = \frac{1}{\sqrt{x+0}+\sqrt{x}} && \text{by the Direct Substitution Property} \\
&\qquad = \frac{1}{2\sqrt{x}}
\end{aligned}$$

5. (10 pts) **Extra credit problem.** Let $f$ be a function and $b$ a real number. Use the delta-epsilon definition of the limit to prove that if $\lim_{x\to b} f(x) = L$ for some real number $L$, then $\lim_{x\to b^-} f(x)$ and $\lim_{x\to b^+} f(x)$ both exist and
$$\lim_{x\to b^-} f(x) = \lim_{x\to b^+} f(x) = L.$$

Let $f$ be a function and $b \in \mathbb{R}$ such that
$$\lim_{x\to b} f(x) = L.$$
To prove that
$$\lim_{x\to b^+} f(x) = L,$$
we need to show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < x - b < \delta$ then $|f(x) - L| < \epsilon$. Since $\lim_{x\to b} f(x) = L$, we can choose a $\delta > 0$ such that if $0 < |x - b| < \delta$ then $|f(x) - L| < \epsilon$ . We can use the same $\delta$ to say that if $0 < x - b < \delta$ then it is also true that $0 < |x - b| < \delta$ and hence $|f(x) - L| < \epsilon$. Therefore $\lim_{x\to b^+} f(x) = L$.

Similary, given any $\epsilon > 0$, we can choose a $\delta > 0$ such that if $0 < |x - b| < \delta$ then $|f(x) - L| < \epsilon$ since $\lim_{x\to b} f(x) = L$. Now, if $0 < b - x < \delta$ then it is also true that $0 < |b - x| < \delta$. But $|b - x| = |x - b|$, so we have $0 < |b - x| < \delta$ and hence $|f(x) - L| < \epsilon$. Therefore $\lim_{x\to b^-} f(x) = L$.