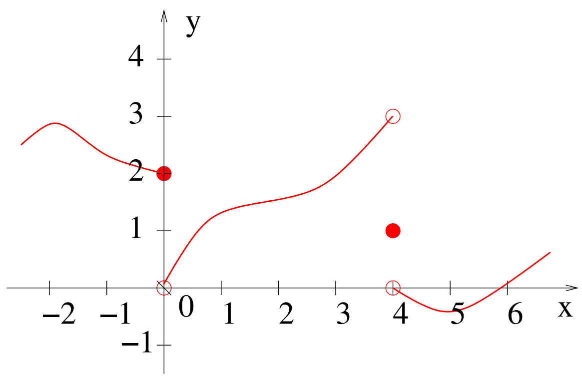
MCS 121 Exam 2 Solutions

1. (10 pts) Sketch the graph of an example of a function f that satisfies all of the conditions

$$\lim_{x\to 0^-} f(x) = 2, \qquad \lim_{x\to 0^+} f(x) = 0, \qquad f(0) = 2,$$
$$\lim_{x\to 4^-} f(x) = 3, \qquad \lim_{x\to 4^+} f(x) = 0, \qquad f(4) = 1.$$

Be sure to explain how you came up with the graph.



Because $f(0) = 2$ and $f(4) = 1$, the points $(0, 2)$ and $(4, 1)$ are on the graph. Since $\lim_{x\to 0^-} f(x) = 2$ the value of f must approach 2 as x approaches 0 from the left. Similarly, $\lim_{x\to 0^+} f(x) = 0$ says that the value of f approaches 0 as x approaches 0 from the right, but $f(0) = 2$, so the point $(0, 0)$ is not part of the graph, which we indicated with an empty circle there. Because $\lim_{x\to 4^-} f(x) = 3$ and $\lim_{x\to 4^+} f(x) = 0$, the graph must also approach the point $(4, 3)$ from the left and $(4, 0)$ from the right. But neither of these points belongs to the graph, which is indicated by empty circles there.

2. (10 pts) Prove

$$\lim_{x\to 0} x^3 = 0$$

using the $\epsilon - \delta$ definition of a limit. Be sure to carefully justify each step in your argument.

We need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $|x^3 - 0| < \epsilon$. We start by figuring out what a good choice for δ is depending on ϵ. We want $|x^3 - 0| < \epsilon$, which is the same thing as

$$|x^3| < \epsilon.$$

Using the fact that $|ab| = |a||b|$ for any real numbers a, b we can write $|x^3| = |x|^3$. So we really want

$$|x|^3 < \epsilon.$$

We can take the cube root of both sides because the cube root function $g(x) = \sqrt[3]{x}$ is an increasing function, so if $|x|^3 < \epsilon$ then

$$\begin{aligned} g\big(|x|^3\big) &< g(\epsilon) \\ \sqrt[3]{|x|^3} &< \sqrt[3]{\epsilon} \\ |x| &< \sqrt[3]{\epsilon} \end{aligned}$$

Since $|x - 0| = |x|$, this suggests that setting $\delta = \sqrt[3]{\epsilon}$ is a good choice for δ. Note that we know $\delta = \sqrt[3]{\epsilon} > 0$ because $\epsilon > 0$. We will now show that if if $0 < |x - 0| < \sqrt[3]{\epsilon}$, then $|x^3 - 0| < \epsilon$. Suppose

$$0 < |x - 0| < \sqrt[3]{\epsilon}.$$

Then

$$|x| < \sqrt[3]{\epsilon}.$$

We can cube both sides because the cube function $h(x) = x^3$ is an increasing function, so if $|x| < \sqrt[3]{\epsilon}$ then

$$\begin{aligned} h\big(|x|\big) &< h\big(\big(\sqrt[3]{\epsilon}\big) \\ |x|^3 &< \sqrt[3]{\epsilon}^3 \\ |x^3| &< \epsilon && \text{because } |x|^3 = |x^3| \\ |x^3 - 0| &< \epsilon \end{aligned}$$

We have shown that for every $\epsilon > 0$, $\delta = \sqrt[3]{\epsilon} > 0$ is such that if $0 < |x - 0| < \delta$ then $|x^3 - 0| < \epsilon$.

3. (10 pts) Find the following limit:

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x^2 - 4x - 45}.$$

Be sure to carefully justify your work by referring to the limit laws or any other theorems you use to evaluate the limit.

First, notice that

$$\lim_{x \to 9} (x^2 - 4x - 45) = 9^2 - 4 \cdot 9 - 45 = 0$$

by direct substitution into the polynomial. Therefore we cannot use LL5. But we can factor $x^2 - 4x - 45$ as

$$x^2 - 4x - 45 = (x - 9)(x + 5).$$

We will simplify the function before we take the limit. We can multiply the numerator and the denominator by $\sqrt{x} + 3$ because $\sqrt{x} + 3 \geq 3$ and so $\sqrt{x} + 3 \neq 0$:

$$\begin{aligned} \frac{\sqrt{x} - 3}{x^2 - 4x - 45} &= \left[\frac{\sqrt{x} - 3}{(x - 9)(x + 5)} \frac{\sqrt{x} + 3}{\sqrt{x} + 3}\right] \\ &= \frac{\sqrt{x}^2 - 3^2}{(x - 9)(x + 5)(\sqrt{x} + 3)} \\ &= \frac{x - 9}{(x - 9)(x + 5)(\sqrt{x} + 3)} \end{aligned}$$

Since $x \neq 9$ as $x \to 9$, we can now cancel the $x-9$ from the fraction:

$$\begin{aligned}
\lim_{x\to 9} \frac{\sqrt{x}-3}{x^2-4x-45} &= \lim_{x\to 9} \frac{1}{(x+5)(\sqrt{x}+3)} \\
&= \lim_{x\to 9} \frac{\lim_{x\to 9} 1}{\lim_{x\to 9}[(x+5)(\sqrt{x}+3)]} && \text{by LL5} \\
&= \lim_{x\to 9} \frac{1}{[\lim_{x\to 9}(x+5)][\lim_{x\to 9}(\sqrt{x}+3)]} && \text{by LL7 and LL4} \\
&= \lim_{x\to 9} \frac{1}{(9+5)\lim_{x\to 9}(\sqrt{x}+3)} && \text{by direct substitution into the polynomial} \\
&= \lim_{x\to 9} \frac{1}{14(\lim_{x\to 9}\sqrt{x}+\lim_{x\to 9} 3)} && \text{by LL1} \\
&= \frac{1}{14(\sqrt{9}+3)} && \text{by LL7 and LL10} \\
&= \frac{1}{84}
\end{aligned}$$

4. (10 pts) Use the formal definiton of the limit (the $\delta - \epsilon$ definition) to prove that

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exist.

Hint: We did this in class by using an indirect argument. We assumed that the limit was L and showed that for $\epsilon = 1$, there exists no $\delta > 0$ that would satisfy the definition of $\lim_{x\to 0} \frac{|x|}{x} = L$.

Suppose the limit did exist and were equal to L. We will show that whatever number L is, for $\epsilon = 1$ there cannot exist a $\delta > 0$ such that if

$$0 < |x-0| < \delta$$

then

$$\left| \frac{|x|}{x} - L \right| < \epsilon.$$

Let $f(x) = |x|/x$. Note that $f(x) = -1$ if x is negative and $f(x) = 1$ if x is positive. We want

$$\left| \frac{|x|}{x} - L \right| < 1,$$

that is

$$L - 1 < \frac{|x|}{x} < L + 1$$

whenever

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0.$$

But no matter how small we make $\delta > 0$, there will always be both positive and negative values of x that satisfy

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0.$$

For example, $x_1 = -\delta/2$ is a negative number that satisfies

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0,$$

and $x_2 = \delta/2$ is a positive number that does. Now, $f(x_1) = -1$ and $f(x_2) = 1$, and -1 and 1 are two units apart on the number line, so they cannot both fit inside the open interval

$(L-1, L+1)$. Here is a more algebraic argument to show this. Since x_1 is negative, we would have to have

$$L - 1 < \frac{|x_1|}{x_1} < L + 1 \implies L - 1 < -1 < L + 1 \implies L - 1 < -1 \implies L < 0.$$

On the other hand, x_2 is positive, and so we would have to have

$$L - 1 < \frac{|x_2|}{x_2} < L + 1 \implies L - 1 < 1 < L + 1 \implies 1 < L + 1 \implies 0 < L.$$

But L cannot be both less than and greater than 0.

5. (5 pts each) **Extra credit problem.** Let f and g be functions of real numbers and $a \in \mathbb{R}$. Suppose that

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

exists.

(a) Suppose $\lim_{x \to a} g(x) = 0$. Show that $\lim_{x \to a} f(x) = 0$.

Let

$$L = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

Notice that for this limit to be L, the value of $f(x)/g(x)$ must be a number close to L when x is close to a but not equal to a. It follows that $g(x)$ cannot be 0 when x is close to a, otherwise the value of $f(x)/g(x)$ would not exist. Hence

$$f(x) = g(x)\,\frac{f(x)}{g(x)}$$

for all values of x near a. Therefore

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left[g(x) \frac{f(x)}{g(x)} \right] \\
&= \left[\lim_{x \to a} g(x) \right] \left[\lim_{x \to a} \frac{f(x)}{g(x)} \right] && \text{by LL5} \\
&= 0 \cdot L \\
&= 0.
\end{aligned}$$

(b) Suppose $\lim_{x \to a} f(x) = 0$. Does $\lim_{x \to a} g(x)$ have to be 0? If you think it does, prove that it does; if you do not think that it does, give a counterexample.

No, the limit of g does not have to be 0, and it is easy enough to find a counterexample. Let $f(x) = 0$, $g(x) = 1$, and $a = 0$. By LL7,

$$\begin{aligned}
\lim_{x \to 0} f(x) &= \lim_{x \to 0} 0 = 0 \\
\lim_{x \to 0} g(x) &= \lim_{x \to 0} 1 = 1 \\
\lim_{x \to 0} \frac{f(x)}{g(x)} &= \lim_{x \to 0} \frac{0}{1} = \lim_{x \to 0} 0 = 0.
\end{aligned}$$