MCS 121 Final Exam Solutions

1. (10 pts) Let

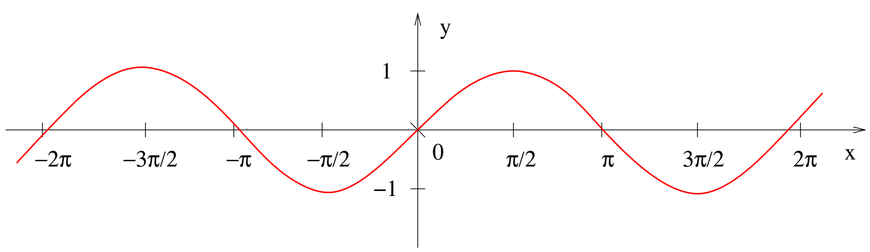
$$f(x) = \frac{x}{1+x} \quad \text{and} \quad g(x) = \sin(2x).$$

Find $f \circ g$ and the largest possible subset of the real numbers that could be the domain of $f \circ g$. Remember to justify your answer.

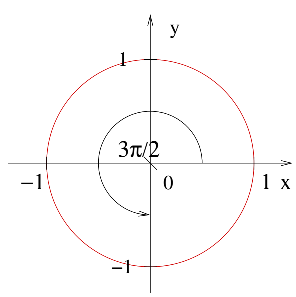
The composition $f \circ g$ is

$$f \circ g(x) = f(g(x)) = f(\sin(2x)) = \frac{\sin(2x)}{1+\sin(2x)}.$$

To find the domain, note that $\sin(2x)$ exists for any real number x because the domain of the sine function is $\mathbb{R}$. But the division can only be done if $1+\sin(2x) \neq 0$. For what values of x is $1+\sin(2x) = 0$? This happens whenever $\sin(2x) = -1$. Looking at the graph of the sine function



or the unit circle



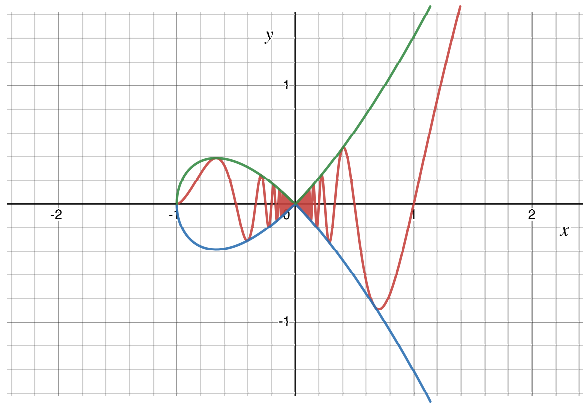
shows that $\sin(2x) = -1$ whenever $2x = \frac{3\pi}{2} + k2\pi$ where $k \in \mathbb{Z}$. Divide by 2 to get $x = \frac{3\pi}{4} + k\pi$. These are the values of x that makes the denominator of $f \circ g(x)$ equal 0, so they must be excluded from the domain. Other real numbers do not cause any trouble. So

$$D(f \circ g) = \{x \in \mathbb{R} \mid x = \frac{3\pi}{4} + k\pi \text{ where } k \text{ is any integer}\}.$$

2. (10 pts) Use the Squeeze Theorem to show that

$$\lim_{x \to 0} \left[\sqrt{x^3 + x^2} \sin\left(\frac{\pi}{x}\right) \right] = 0.$$

Hint: Here is the graph of this function to help you find two functions such that $f(x) = \sqrt{x^3+x^2}\sin(\pi/x)$ is squeezed between them for values of x close to 0 (how close exactly?).



Let $f(x) = \sqrt{x^3 + x^2}\sin(\pi/x)$. It is clear that $x = 0$ is not in the domain of g since we would have a 0 in the denominator if $x = 0$. Since we are interested in $x \to 0$, we actually know $x \neq 0$ in this context. Let $g(x) = -\sqrt{x^3 + x^2}$ and $h(x) = \sqrt{x^3 + x^2}$. The diagram above includes the graphs of g and h in blue and green respectively. The graph of f is red. Note that

$$x^3 + x^2 = x^2(x + 1).$$

If x is close to 0, but not equal to 0, x^2 is positive. So is $x + 1$. In fact $x + 1 > 0$ for all $x > -1$. So $x^3 + x^2 > 0$ if $x > -1$ and $x \neq 0$. Therefore $\sqrt{x^3 + x^2} > 0$ if $x > -1$ and $x \neq 0$.

We know

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

for all real numbers $x \neq 0$. Since we are interested in what happens as $x \to 0$, we may assume $x > 1$ and $x \neq 0$. We can now multiply all sides of the inequality above by the positive number $\sqrt{x^3 + x^2}$:

$$\underbrace{-\sqrt{x^3 + x^2}}_{g(x)} \leq \underbrace{\sqrt{x^3 + x^2}\sin\left(\frac{\pi}{x}\right)}_{f(x)} \leq \underbrace{\sqrt{x^3 + x^2}}_{h(x)}.$$

That is $f(x)$ is squeezed between $g(x)$ and $h(x)$ for all $x \neq 0$ near 0.

Now,

$$\begin{aligned} \lim_{x \to 0} \sqrt{x^3 + x^2} &= \sqrt{\lim_{x \to 0}(x^3 + x^2))} && \text{by LL11} \\ &= \sqrt{0^3 + 0^2} && \text{by direct substitution into the polynomial} \\ &= 0 \end{aligned}$$

Now it follows by LL3 that

$$\lim_{x \to 0} \sqrt{x^3 + x^2} = -\lim_{x \to 0} \sqrt{x^3 + x^2} = 0.$$

Therefore

$$\lim_{x \to 0} \left[\sqrt{x^3 + x^2}\sin\left(\frac{\pi}{x}\right)\right] = 0$$

by the Squeeze Theorem.

3. (10 pts) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

Since $y'(x) = 2x+1$, the slope of the tangent line at some point a is $y'(a) = 2a+1$. Hence the equation of the tangent line is $y = (2a+1)x+b$. We can find b by substituting the point (a, a^2+a) into the equation of the line:

$$a^2 + a = (2a+1)a + b \implies b = a^2 + a - (2a+1)a = -a^2.$$

So the equation of the tangent line at $x = a$ is $y = (2a+1)x - a^2$. This line passes through $(2, -3)$ if $(2, -3)$ satisfies the equation:

$$-3 = (2a+1)2 - a^2 \iff a^2 - 4a - 5 = 0 \iff (a-5)(a+1) = 0$$

So either $a = 5$ or $a = -1$, and the equations of the two tangent lines that pass through $(2, -3)$ are

$$y = 11x - 25 \qquad \text{and} \qquad y = -x - 1.$$

4. (10 pts) Use the formal definition of the limit (that is ϵ and N) to show that

$$\lim_{x\to\infty} \frac{1}{x^3} = 0.$$

We need to show that for all $\epsilon > 0$ there is exists some corresponding number N such that if $x > N$, then

$$\left|\frac{1}{x^3} - 0\right| < \epsilon.$$

Suppose $\epsilon > 0$. We want to make

$$\left|\frac{1}{x^3} - 0\right| < \epsilon$$

true. Note that $x \to \infty$, so we can assume x is positive. Therefore $1/x^3 - 0 = 1/x^3$ is a positive number. So we do not need the absolute value. Since x^3 and ϵ are boht positive,

$$\frac{1}{x^3} < \epsilon \implies 1 < \epsilon x^3 \implies \frac{1}{\epsilon} < x^3.$$

The cube root function $f(x) = \sqrt[3]{x}$ is increasing on all real numbers, so we can take cube roots of both sides:

$$\frac{1}{\epsilon} < x^3 \implies \sqrt[3]{\frac{1}{\epsilon}} < \sqrt[3]{x^3} = x \implies \frac{1}{\sqrt[3]{\epsilon}} < x.$$

This suggests that we should set $N = 1/\sqrt[3]{\epsilon}$. We will now show that if $x > N = 1/\sqrt[3]{\epsilon}$, then

$$\left|\frac{1}{x^3} - 0\right| < \epsilon.$$

Suppose $x > 1/\sqrt[3]{\epsilon}$. Since the cube function $g(x) = x^3$ is increasing on all real numbers, we can cube both sides:

$$\frac{1}{\sqrt[3]{\epsilon}} < x \implies \left(\frac{1}{\sqrt[3]{\epsilon}}\right)^3 < x^3.$$

Since x^3 and ϵ are both positive,

$$\frac{1}{\epsilon} < x^3 \implies 1 < \epsilon x^3 \implies \frac{1}{x^3} < \epsilon.$$

It follows that

$$\left|\frac{1}{x^3} - 0\right| = \frac{1}{x^3} < \epsilon.$$

5. (10 pts) Use the definition of the derivative to find

$$\frac{d}{dx}\frac{1}{\sqrt{x}+3}.$$

Be sure to carefully justify every step of your calculation.

Let $f(x) = 1/(\sqrt{x}+3)$. Then

$$f'(a) = \lim_{x\to a}\frac{\frac{1}{\sqrt{x}+3} - \frac{1}{\sqrt{a}+3}}{x-a} = \lim_{x\to a}\frac{\frac{\sqrt{a}+3-(\sqrt{x}+3)}{(\sqrt{x}+3)(\sqrt{a}+3)}}{x-a} = \lim_{x\to a}\frac{\sqrt{a}-\sqrt{x}}{(x-a)(\sqrt{x}+3)(\sqrt{a}+3)}.$$

We will want to multiply and divide by $\sqrt{a}+\sqrt{x}$, but we can only do that if we know it is not 0. Since $\sqrt{a} \geq 0$ and $\sqrt{x} \geq 0$, the only way $\sqrt{a}+\sqrt{x}$ could be 0 is if $x = 0$ and $a = 0$. But we know $x \neq a$ as $x \to a$. So for sure, $\sqrt{a}+\sqrt{x} \neq 0$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{\sqrt{a}-\sqrt{x}}{(x-a)(\sqrt{x}+3)(\sqrt{a}+3)} &= \lim_{x\to a}\frac{(\sqrt{a}-\sqrt{x})(\sqrt{a}+\sqrt{x})}{(\sqrt{a}+\sqrt{x})(x-a)(\sqrt{x}+3)(\sqrt{a}+3)} \\
&= \lim_{x\to a}\frac{a-x}{(\sqrt{a}+\sqrt{x})(x-a)(\sqrt{x}+3)(\sqrt{a}+3)} \\
&= \lim_{x\to a}\frac{-1}{(\sqrt{a}+\sqrt{x})(\sqrt{x}+3)(\sqrt{a}+3)} \\
&= \frac{\lim_{x\to a}(-1)}{\lim_{x\to a}\left[(\sqrt{a}+\sqrt{x})(\sqrt{x}+3)(\sqrt{a}+3)\right]} && \text{by LL5} \\
&= \frac{-1}{\lim_{x\to a}(\sqrt{a}+\sqrt{x})\lim_{x\to a}(\sqrt{x}+3)\lim_{x\to a}(\sqrt{a}+3)} && \text{by LL7 and LL4} \\
&= -\frac{1}{(\lim_{x\to a}\sqrt{a}+\lim_{x\to a}\sqrt{x})(\lim_{x\to a}\sqrt{x}+\lim_{x\to a}3)(\sqrt{a}+3)} && \text{by LL1 and LL7} \\
&= -\frac{1}{(\sqrt{a}+\sqrt{a})(\sqrt{a}+3)(\sqrt{a}+3)} && \text{by LL7 and LL10} \\
&= -\frac{1}{2\sqrt{a})(\sqrt{a}+3)^2}.
\end{aligned}$$

We can conclude

$$\frac{d}{dx}\frac{1}{\sqrt{x}+3} = -\frac{1}{2\sqrt{x})(\sqrt{x}+3)^2}.$$

Note that you can easily check your answer here by finding the derivative using the chain rule.

6. (5 pts each) Let $f(x) = \csc(x) = 1/\sin(x)$.
   (a) Find the critical points of f on the interval $[0, 2\pi]$.

   Let us find f':

$$f'(x) = [\sin(x)]^{-1} = (-1)[\sin(x)]^{-2}\cos(x) = -\frac{\cos(x)}{\sin^2(x)}.$$

   by the Chain Rule. We also could have used the Quotient Rule to differentiate f. We are looking for values of x where $f'(x) = 0$ or does not exist. It immediately stands out that $f'(x)$ does not exist if $\sin^2(x) = 0$, which true whenever $\sin(x) = 0$. But if $\sin(x) = 0$, then $f(x) = 1/\sin(x)$ does not have a value either. So any place where the denominator of $f'(x)$ is 0 also does not belong to the domain of f. In fact, if x is any number such that $\sin(x) \neq 0$, then $f'(x)$ has a legitimate value. That is f is differentiable at every x in its domain. So the only kind of critical point it can have is where $f'(x) = 0$. It is clear that

$$0 = f'(x) = -\frac{\cos(x)}{\sin^2(x)}$$

when $\cos(x) = 0$ and $\sin(x) \neq 0$. The only numbers in $[0, 2\pi]$ where $\cos(x) = 0$ are $\pi/2$ and $3\pi/2$. In fact,

$$f'\left(\frac{\pi}{2}\right) = -\frac{\cos(\pi/2)}{\sin^2(\pi/2)} = -\frac{0}{1^2} = 0$$
$$f'\left(\frac{3\pi}{2}\right) = -\frac{\cos(3\pi/2)}{\sin^2(3\pi/2)} = -\frac{0}{(-1)^2} = 0.$$

So the critical points of f on $[0, 2\pi]$ are $\pi/2$ and $3\pi/2$.

(b) Use either f' or f'' to decide which of the critical points you found in part (a) are local minima, which ones are local maxima, and which ones are neither.

For x close to $\pi/2$, the value of $\cos(x)$ is positive if $x < \pi/2$ and negative if $x > \pi/2$. The value of $\sin^2(x)$ is some number close to 1. So $f'(x) = -\cos(x)/\sin^2(x)$ switches sign from negative to positive at $\pi/2$. Hence f has a local minimum at $\pi/2$.
For x close to $3\pi/2$, the value of $\cos(x)$ is negative if $x < 3\pi/2$ and positive if $x > 3\pi/2$. The value of $\sin^2(x)$ is some number close to 1. So $f'(x) = -\cos(x)/\sin^2(x)$ switches sign from positive to negative at $3\pi/2$. Hence f has a local maximum at $3\pi/2$.

Alternately, we can find f'' using the Quotient Rule and the Chain Rule:

$$\begin{aligned}
f''(x) &= -\frac{\left(\frac{d}{dx}\cos(x)\right)\sin^2(x) - \cos(x)\frac{d}{dx}\sin^2(x)}{\sin^4(x)} \\
&= -\frac{-\sin(x)\sin^2(x) - \cos(x)2\sin(x)\cos(x)}{\sin^4(x)} \\
&= \frac{\sin(x)\left(\sin^2(x) + 2\cos^2(x)\right)}{\sin^4(x)} \\
&= \frac{\sin^2(x) + 2\cos^2(x)}{\sin^3(x)} \\
&= \frac{\overbrace{\sin^2(x) + \cos^2(x)}^{1} + \cos^2(x)}{\sin^3(x)} \\
&= \frac{1 + \cos^2(x)}{\sin^3(x)}.
\end{aligned}$$

Now,

$$f''\left(\frac{\pi}{2}\right) = \frac{1 + \cos^2(\pi/2)}{\sin^3(\pi/2)} = \frac{1 + 0^2}{1^3} = 1$$
$$f''\left(\frac{3\pi}{2}\right) = \frac{1 + \cos^2(3\pi/2)}{\sin^3(3\pi/2)} = \frac{1 + 0^2}{(-1)^3} = -1.$$

So by the Second Derivative Test, f has a local minimum at $\pi/2$ and a local maximum at $3\pi/2$.

7. **Extra credit problem.**

(a) (8 pts) In a contest between two rival monasteries, two Tibetan monks leave their monasteries at 7:00 AM to climb the same mountain. The two monasteries are exactly the same distance from the peak. The contest is a tie when both monks reach the peak at exactly 7:00 PM. Show that there was a point in time during the day when both monks were walking at exactly the same velocity.

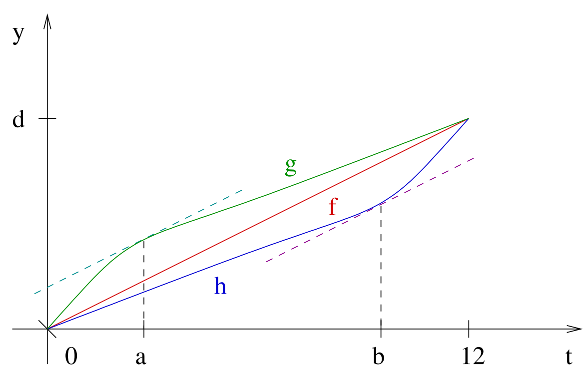
Hint: Verify that the difference of the position functions satisfies the conditions of Rolle's Theorem or the Mean Value Theorem, then apply the theorem.

Let t be time measured in hours after 7:00 AM. Let $f(t)$ be the first monk's position along the path measured from his starting point at the bottom of the mountain. Let g(t) be the second monk's position along the path measured similarly. Then we know $f(0) = g(0) = 0$ and $f(12) = g(12) = d$ where d is the distance of the peak from the monasteries along the paths the monks take.

Note that f and g must be continuous functions of time, as monks are subject to the laws of physics and therefore their position must vary continuously with time (e.g. monks cannot teleport and just show up in a different place from one moment to the next.) Also, both f and g must be differentiable functions because the monks must always be walking at some velocity, and velocity is the derivative of position with respect to time. Now, let $h(t) = f(t) - g(t)$. Since f and g are continuous and differentiable, so is h. We know $h(0) = f(0) - g(0) = 0$ and $h(12) = f(12) - g(12) = 0$. Therefore h satisfies the conditions of Rolle's Theorem over the interval $[0, 12]$. By Rolle's Theorem, there must be some $c \in (0, 12)$ where $0 = h'(c) = f'(c) - g'(c)$. So $f'(c) = g'(c)$. That is the monks are moving at the same velocity at time c.

(b) (7 pts) Suppose three monks from three monasteries all the same distance from the peak compete to climb the mountain. All three leave at 7:00 AM and the race is tied when they all arrive at 7:00 PM. Does there have to be a time when all three are walking at the same velocity? If you think there must be such a point in time, prove it; if you do not think there is, find an argument or counterexample to show it.

There does not have to be such a time. Let $f(t)$, $g(t)$, and $h(t)$ be the positions of the three monks as a function of time. Consider the following three graphs:



Notice that the first monk walks at a constant rate. The second monk initially walks fast then slows down. The only time when his velocity is the same as the first monk's is at time a. The third monk initially walks slowly then speeds up. The only time when his velocity is the same as the first monk's is at time b. So there is no point in time when both the second and third monks move at the same velocity as the first monk. So at no time do all three monks walk at the same velocity.