MCS 121 Exam 1 Solutions

1. (5 pts each) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.
   (a) If f and g are both odd functions, is the product fg always odd? If you think it is, prove that it always is; if you do not think it is, find a counterexample.

   The product fg is not always odd. Here is a counterexample:

   $$f(x) = x$$
   $$g(x) = x^3$$
   $$(fg)(x) = x \cdot x^3 = x^4$$

   It is easy to see that f and g are odd functions, since

   $$f(-x) = -x = -f(x)$$
   $$g(-x) = (-x)^3 = -x^3 = -g(x)$$

   for all $x \in \mathbb{R}$. But $(fg)(-x) \neq -(fg)(x)$ in general. For example,

   $$(fg)(-2) = (-2)^4 = 16 \neq -16 = -2^4 = -(fg)(2).$$

   You did not have to do this, but in fact, we can easily show that fg is always an even function:

   $$(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$$

   for all $x \in \mathbb{R}$. Note that while finding that fg must always be an even function gives us a hint that it may not be odd, it does not prove that. A function can be both even and odd (well, at least in one special case), so knowing that fg is even does not tell us that fg cannot be odd.

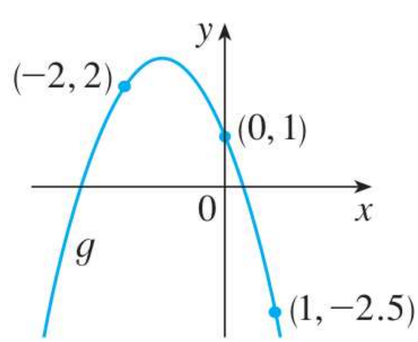
   (b) What if f is even and g is odd? Does fg have to be odd then? If you think it does, prove that it does; if you do not think it does, find a counterexample.

   We know $f(-x) = f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. So

   $$(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -(fg)(x)$$

   for all $x \in \mathbb{R}$. This shows fg is an odd function.

2. (10 pts) Find an expression for the quadratic polynomial function whose graph is shown.



We know g is a quadratic polynomial, so it must be of the form $g(x) = ax^2 + bx + c$. Since the graph passes through $(-2, 2)$, $(0, 1)$, and $(1, -2.5)$,

$$\begin{aligned} 2 &= g(-2) = a(-2)^2 + b(-2) + c = 4a - 2b + c \\ 1 &= g(0) = a \cdot 0^2 + b \cdot 0 + c = c \\ -2.5 &= g(1) = a \cdot 1^2 + b \cdot 1 + c = a + b + c \end{aligned}$$

The second equation tells us $c = 1$. Substituting this into the other equations yields

$$\begin{aligned} 4a - 2b + 1 = 2 &\implies 4a - 2b = 1 \\ a + b + 1 = -2.5 &\implies a + b = -3.5. \end{aligned}$$

Multiply the second equation by 2 to get $2a + 2b = -7$, then add the two equations to get $6a = -6$, and hence $a = -1$. Now $b = -3.5 - a = -2.5$. So

$$g(x) = -x^2 - 2.5x + 1.$$

We can easily check that this is correct by substituting the three points:

$$\begin{aligned} g(-2) &= -(-2)^2 - 2.5(-2) + 1 = 2 && \checkmark \\ g(0) &= -0^2 - 2.5 \cdot 0 + 1 = 1 && \checkmark \\ g(1) &= -1^2 - 2.5 \cdot 1 + 1 = -2.5 && \checkmark \end{aligned}$$

3. (10 pts) Let

$$f(x) = \frac{1}{1 - |x|} \qquad \text{and} \qquad g(x) = \cos(x).$$

Find $f \circ g(x)$ and the largest possible subset of the real numbers that could be the domain of $f \circ g$.

First,

$$f \circ g(x) = f(g(x)) = \frac{1}{1 - |\cos(x)|}.$$

As for the domain, we need to avoid values of x that make the denominator 0:

$$1 - |\cos(x)| = 0 \iff |\cos(x)| = 1 \iff \cos(x) = \pm 1.$$

Thinking either about the unit circle or the graph of the cosine function, it is clear that $\cos(x) = 1$ when $x = 2k\pi$ for any integer k, and $\cos(x) = -1$ when $x = \pi + 2k\pi$ for any integer k. Combining these gives $x = k\pi$ for any integer k. For any other value of k, the denominator $1 - |\cos(x)|$ is not 0, and hence $f \circ g(x)$ is a real number. Therefore

$$D(f \circ g) = \{x \in \mathbb{R} \mid x \neq k\pi \text{ for any } k \in \mathbb{Z}\}.$$

4. Beavis and Butthead play frog baseball. As Beavis pitches the (poor, unlucky) frog, Butthead observes that the position of the frog measured as its distance in feet from Beavis is given by the function

$$f(t) = 90t - 13t^2$$

where t is time in seconds and $t = 0$ is the moment Beavis releases the frog.

(a) (7 pts) Find the average velocity of the frog over the following time intervals:

$$[3, 4], [3, 3.5], [3, 3.1], [2, 3], [2.5, 3], [2.9, 3].$$

What are the physical units?

Using

$$v_{\text{ave}} = \frac{f(t_1) - f(t_2)}{t_1 - t_2}$$

we get the following values:

Interval	Average velocity
$[3,4]$	$\frac{f(4)-f(3)}{4-3}=\frac{152-153}{1}=-1$
$[3,3.5]$	$\frac{f(3.5)-f(3)}{3.1-3}=\frac{155.75-153}{0.5}=5.5$
$[3,3.1]$	$\frac{f(3.1)-f(3)}{3.1-3}=\frac{154.07-153}{0.1}=10.7$
$[2,3]$	$\frac{f(3)-f(2)}{3-2}=\frac{153-128}{1}=25$
$[2.5,3]$	$\frac{f(3)-f(2.5)}{3-2.5}=\frac{153-143.75}{0.5}=18.5$
$[2.9,3]$	$\frac{f(3)-f(2.9)}{3-2.9}=\frac{153-151.67}{0.1}=13.3$

The physical units are ft/s, since the numerator is measured in feet and the denominator in seconds.

(b) (3 pts) Use your results in part (a) to estimate the instantaneous velocity of the frog at time $t=3$. Does your result seem realistic to you?

The results in part (a) do not give us a really good way to estimate the instantaneous velocity accurately. All we can say is that the actual value is likely to be somewhere between 10.7 ft/s and 13.3 ft/s. As the time interval gets shorter, the average velocity should be getting closer and closer to the instantaneous velocity. The trend is that for a time interval preceding $t=3$, the average velocity seems to be decreasing toward the actual value of the instantaneous veloctiy, while for a time interval exceeding $t=3$, the average velocity seems to be increasing toward the actual value of the instantaneous velocity. Therefore it makes sense to expect that the actual value is likely to be somewhere between 10.7 ft/s and 13.3 ft/s. The arithmetic mean $\frac{10.7+13.3}{2}=12$ ft/s would be a reasonable guess. If we wanted to have a more accurate estimate, we could continue calculating the average velocity over shorter and shorter time intervals until we have a better idea of the number that those average velocities as approaching. Note that 12 ft/s is

$$\frac{12\cdot 3600}{5280}\approx 8.2 \text{ miles/h},$$

which seems reasonable enough for the speed of a flying frog, which is not really aerodynamic, 3 seconds after it was pitched by a skinny teenager.

5. (10 pts) **Extra credit problem.** Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that f is odd and f is decreasing on the interval $[0,\infty)$. Does f have to be increasing, or decreasing, or can it be neither on the interval $(-\infty,0]$? Find a rigorous argument using the definition of odd functions and increasing/decreasing functions to justify your answer.

Since f is odd, its graph must be symmetric across the origin. Visualizing this for a particular example (e.g. you could try $f(x)=-x^3$) gives us the idea that f is probably also decreasing on $(-\infty,0]$. To show this rigorously, we would have to show that if x_1 and x_2 are numbers such that $x_1<x_2\le 0$, then $f(x_1)>f(x_2)$. So let x_1 and x_2 be such numbers. Since x_1 and x_2 are nonpositive, $-x_1$ and $-x_2$ must be nonnegative. That is $0\le -x_1,-x_2$. In fact, we also know that $-x_2<-x_1$, which follows by multiplying the inequality $x_1<x_2$ by -1. Since f is decreasing on the interval $[0,\infty)$, and $0\le -x_2<-x_1$, it must be true that $f(-x_2)>f(-x_1)$. But f is odd, so $f(-x_2)=-f(x_2)$ and $f(-x_1)=-f(x_1)$. Hence $-f(x_2)>-f(x_1)$. Multiplying this inequality by -1 yields $f(x_2)<f(x_1)$, which is what we wanted to prove.