MCS 121 Exam 2

1. (10 pts) Prove
$$\lim_{x\to -2}(3x+5) = -1$$
using the $\epsilon - \delta$ definition of a limit.

   We need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that if
$$0 < |x-(-2)| < \delta$$
then
$$|f(x) - (-1)| < \epsilon.$$
So we want
$$|(3x+5) - (-1)| < \epsilon.$$
Notice that
$$|(3x+5) - (-1)| = |3x+6| = |3(x+2)| = |3||x+2| = 3|x+2|.$$
This suggests that we should make $3|x+2| < \epsilon$, that is $|x+2| < \epsilon/3$.
   So let $\delta = \epsilon/3$. If $0 < |x+2| < \delta = \epsilon/3$, then
$$|(3x+5) - (-1)| = |3x+6| = |3(x+2)| = |3||x+2| = 3|x+2| < 3\frac{\epsilon}{3} = \epsilon,$$
which is what we wanted to show.

2. (10 pts) Find the exact value of
$$\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}.$$
Be sure to justify every step of your calculation by referring to the Limit Laws and any other result you use.

   We cannot start by using Limit Law #5 because we will end up with a limit in the denominator that is 0. So we will multiply the numerator and the denominator by $\sqrt{3+x}+\sqrt{3}$. This is only legitimate if we know $\sqrt{3+x}+\sqrt{3} \neq 0$. Since $x \to 0$, x will be a number in some sufficiently small neighborhood of 0. If that neighborhood is small enough (and we can always make it small enough), then $3+x \geq 0$. In fact all we need for this is that $x \geq -3$, which will certainly be true if $|x-3| < 3$. Then
$$\sqrt{3+x}+\sqrt{3} \geq 0 + \sqrt{3} > 0,$$
and so $\sqrt{3+x}+\sqrt{3} \neq 0$. Now,
$$\begin{aligned}\frac{\sqrt{3+x}-\sqrt{3}}{x} &= \frac{\sqrt{3+x}-\sqrt{3}}{x}\,\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\\ &= \frac{\sqrt{3+x}^2-\sqrt{3}^2}{x\left(\sqrt{3+x}+\sqrt{3}\right)}\\ &= \frac{3+x-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}\\ &= \frac{x}{x\left(\sqrt{3+x}+\sqrt{3}\right)}\end{aligned}$$

We can now cancel x because $x \to 0$ means x is a number close to 0 but not equal to 0. So

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x} &= \lim_{x\to 0} \frac{1}{\sqrt{3+x}+\sqrt{3}} \\
&= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\left(\sqrt{3+x}+\sqrt{3}\right)} && \text{by LL5} \\
&= \frac{1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}} && \text{by LL7 and LL1} \\
&= \frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}} && \text{by LL11 and LL7} \\
&= \frac{1}{\sqrt{3+0}+\sqrt{3}} && \text{by direct substitution into the polynomial} \\
&= \frac{1}{2\sqrt{3}}.
\end{aligned}$$

3. (10 pts) Let

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

be a polynomial with real coefficients and let a be any real number. Prove that p has the direct substitution property

$$\lim_{x\to a} p(x) = p(a).$$

You may use the Limit Laws, but make sure you carefully justify each step of your argument.

Here is how we did this in class:

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0) \\
&= \lim_{x\to a}(c_n x^n) + \lim_{x\to a}(c_{n-1}x^{n-1}) + \cdots + \lim_{x\to a}(c_1 x) + \lim_{x\to a} c_0 && \text{by LL1} \\
&= c_n \lim_{x\to a} x^n + c_{n-1}\lim_{x\to a} x^{n-1} + \cdots + c_1 \lim_{x\to a} x + \lim_{x\to a} c_0 && \text{by LL3} \\
&= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 && \text{by LL9 and LL7} \\
&= f(a).
\end{aligned}$$

4. (10 pts) Use the Squeeze Theorem to evaluate

$$\lim_{x\to 0}\left[x^3 \cos\left(\frac{1}{x}\right)\right].$$

Let us start by noting that

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$

for any real number $x \neq 0$. First, suppose $x > 0$. Then x^3 is also positive, hence we can multiply the inequality above by x^3, and we get

$$-x^3 \le x^3 \cos\left(\frac{1}{x}\right) \le x^3.$$

Now,

$$\lim_{x\to 0^+} x^3 = 0^3 = 0$$

by LL9 or by the direct subtitution property of polynomials. By LL3,

$$\lim_{x\to 0^+} (-x^3) = -\lim_{x\to 0^+} x^3 = -0^3 = 0.$$

Therefore

$$\lim_{x\to 0^+} \left[x^3 \cos\left(\frac{1}{x}\right)\right] = 0$$

by the Squeeze Theorem.

Now, if $x < 0$ then x^3 is negative, and so

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1 \implies -x^3 \ge x^3 \cos\left(\frac{1}{x}\right) \ge x^3.$$

Since

$$\lim_{x\to 0^-} x^3 = 0^3 = 0$$

by LL9 or by the direct subtitution property of polynomials, and

$$\lim_{x\to 0^-} (-x^3) = -\lim_{x\to 0^-} x^3 = -0^3 = 0,$$

we can conclude by the Squeeze Theorem that

$$\lim_{x\to 0^-} \left[x^3 \cos\left(\frac{1}{x}\right)\right] = 0.$$

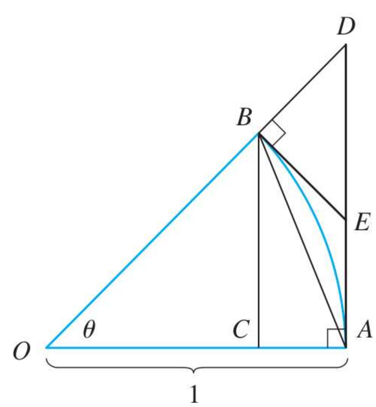
Finally, since the left and the right limits are both 0,

$$\lim_{x\to 0} \left[x^3 \cos\left(\frac{1}{x}\right)\right] = 0.$$

5. **Extra credit problem.**

(a) (7 pts) Use the diagram below to show that if $0 < \theta < \pi/2$, then

$$1 - \theta \le \cos(\theta) \le 1.$$



Suppose $0 < \theta < \pi/2$. We start by noticing that

$$0 \le AC$$

because AC is the side of a triangle. In fact, $\triangle ABC$ is a right triangle. Since AB is the hypotenuse, it is the longest side of $\triangle ABC$. Hence

$$AC \le AB.$$

Since AB is also a secant line in the unit circle, it is the shortest path between points a and B. In particular, the arc $\widehat{\mathrm{AB}}$ is at least as long as the line segment AB. But $\widehat{\mathrm{AB}} = \theta$. Hence

$$0 \le AC \le AB \le \widehat{\mathrm{AB}} = \theta.$$

Now,

$$\cos(\theta) = OC = OA - AC = 1 - AC.$$

We know $AC \le \theta$, and so

$$1 \ge 1 - AC \ge 1 - \theta \implies 1 \ge \cos(\theta) \ge 1 - \theta.$$

(b) (3 pts) Now, use the Squeeze Theorem to prove that

$$\lim_{\theta \to 0^+} \cos(\theta) = 1.$$

We know from part (a) that if $0 < \theta < \pi/2$, then

$$1 - \theta \le \cos(\theta) \le 1.$$

Now,

$$\lim_{\theta \to 0^+} 1 = 1$$

by LL7, and

$$\lim_{\theta \to 0^+} (1 - \theta) = 1 - 0 = 1$$

by direct substitution into the polynomial. Hence

$$\lim_{\theta \to 0^+} \cos(\theta) = 1$$

by the Squeeze Theorem.