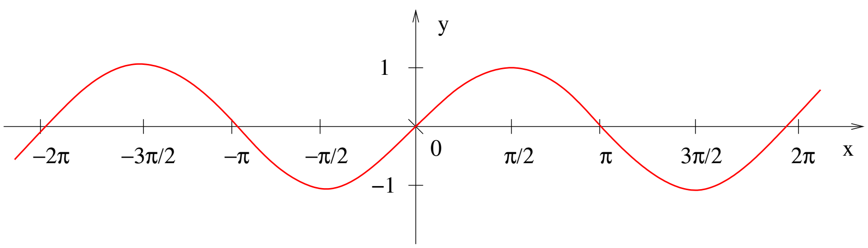
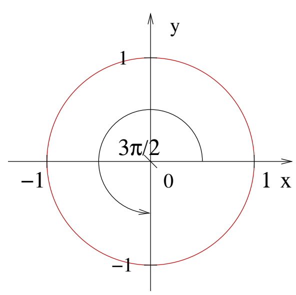
MCS 121 Final Exam Solutions

1. (10 pts) Let
$$f(x) = \frac{x}{1+x} \quad \text{and} \quad g(x) = \sin(2x).$$
Find $f \circ g$ and the largest possible subset of the real numbers that could be the domain of $f \circ g$. Remember to justify your answer.

$$f \circ g(x) = f(g(x)) = f(\sin(2x)) = \frac{\sin(2x)}{1+\sin(2x)}.$$
To find the domain, note that $\sin(2x)$ exists for any real number x because the domain of the sine function is $\mathbb{R}$. But the division can only be done if $1+\sin(2x) \neq 0$. For what values of x is $1+\sin(2x) = 0$? This happens whenever $\sin(2x) = -1$. Looking at the graph of the sine function



or the unit circle



shows that $\sin(2x) = -1$ whenever $2x = \frac{3\pi}{2} + k2\pi$ where $k \in \mathbb{Z}$. Divide by 2 to get $x = \frac{3\pi}{4} + k\pi$. These are the values of x that makes the denominator of $f \circ g(x)$ equal 0, so they must be excluded from the domain. Other real numbers do not cause any trouble. So
$$D(f \circ g) = \{x \in \mathbb{R} \mid x = \frac{3\pi}{4} + k\pi \text{ where } k \text{ is any integer}\}.$$

2. (10 pts) Prove
$$\lim_{x \to 0} x^3 = 0$$
using the $\epsilon - \delta$ definition of a limit. Be sure to carefully justify each step in your argument.

We need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $|x^3 - 0| < \epsilon$. We start by figuring out what a good choice for δ is depending on ϵ. We want $|x^3 - 0| < \epsilon$, which is the same thing as
$$|x^3| < \epsilon.$$
Using the fact that $|ab| = |a||b|$ for any real numbers a, b we can write $|x^3| = |x|^3$. So we really want
$$|x|^3 < \epsilon.$$

We can take the cube root of both sides because the cube root function $g(x) = \sqrt[3]{x}$ is an increasing function, so if $|x|^3 < \epsilon$ then

$$\begin{aligned} g\big(|x|^3\big) &< g(\epsilon) \\ \sqrt[3]{|x|^3} &< \sqrt[3]{\epsilon} \\ |x| &< \sqrt[3]{\epsilon} \end{aligned}$$

Since $|x - 0| = |x|$, this suggests that setting $\delta = \sqrt[3]{\epsilon}$ is a good choice for δ. Note that we know $\delta = \sqrt[3]{\epsilon} > 0$ because $\epsilon > 0$. We will now show that if if $0 < |x - 0| < \sqrt[3]{\epsilon}$, then $|x^3 - 0| < \epsilon$. Suppose

$$0 < |x - 0| < \sqrt[3]{\epsilon}.$$

Then

$$|x| < \sqrt[3]{\epsilon}.$$

We can cube both sides because the cube function $h(x) = x^3$ is an increasing function, so if $|x| < \sqrt[3]{\epsilon}$ then

$$\begin{aligned} h\big(|x|\big) &< h\big(\big(\sqrt[3]{\epsilon}\big) \\ |x|^3 &< \sqrt[3]{\epsilon}^3 \\ |x^3| &< \epsilon && \text{because } |x|^3 = |x^3| \\ |x^3 - 0| &< \epsilon \end{aligned}$$

We have shown that for every $\epsilon > 0$, $\delta = \sqrt[3]{\epsilon} > 0$ is such that if $0 < |x - 0| < \delta$ then $|x^3 - 0| < \epsilon$.

3. (10 pts) Let g be a function of real numbers. If g is twice differentiable and $f(x) = xg(x^2)$, find f'' in terms of g, g', and g''.

   First,

$$\begin{aligned} f'(x) &= \frac{d}{dx}xg(x^2) + x\frac{d}{dx}g(x^2) && \text{by the product rule} \\ &= g(x^2) + xg'(x^2)(2x) && \text{by the chain rule} \\ &= g(x^2) + 2x^2g'(x^2) \end{aligned}$$

   Now,

$$\begin{aligned} f''(x) &= \frac{d}{dx}g(x^2) + \frac{d}{dx}(2x^2)g'(x^2) + 2x^2\frac{d}{dx}g'(x^2) && \text{by the sum and product rules} \\ &= g'(x^2)(2x) + 2(2x)g'(x^2) + 2x^2g''(x^2)(2x) && \text{by the chain rule} \\ &= 6xg(x^2) + 4x^3g''(x^2) \end{aligned}$$

4. (10 pts) Use the definition of the derivative to find the derivative of $f(x) = 1/x^2$. Be sure to carefully justify each step in your calculation.

   Since 0 is obviously not in the domain of f, let $a \neq 0$. Then

$$f'(a) = \lim_{x \to a} \frac{\frac{1}{x^2} - \frac{1}{a^2}}{x - a}$$

Let us work on the expression in the limit:

$$\frac{\frac{a^2}{x^2a^2} - \frac{x^2}{x^2a^2}}{x - a} = \frac{a^2 - x^2}{x^2a^2(x - a)} = \frac{a^2 - x^2}{x^2a^2(x - a)} = \frac{(a + x)(a - x)}{x^2a^2(x - a)}$$

Since $a - x = -(x-a)$, we can cancel the factor $x - a$ as long as it is not 0. But in the limit as $x \to a$, x is close to a but $x \neq a$. So

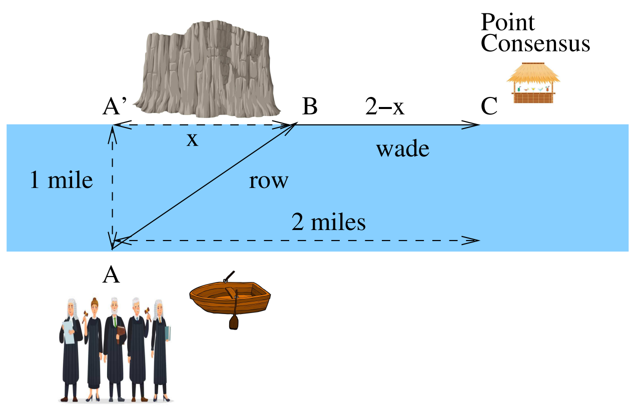
$$\frac{(a+x)(a-x)}{x^2a^2(x-a)} = -\frac{a+x}{x^2a^2}.$$

Therefore

$$f'(a) = \lim_{x\to a}\frac{\frac{1}{x^2}-\frac{1}{a^2}}{x-a} = \lim_{x\to a}\left[-\frac{a+x}{x^2a^2}\right] = -\frac{a+a}{a^2a^2} = -\frac{2a}{a^4} = -\frac{2}{a^3}$$

where we evaluated the limit by direct substitution into the rational function. Hence $f'(x) = -2/x^3$.

5. (10 pts) The Row vs. Wade problem. As part of a team bulding exercise, the nine justices of the Supreme Court of the United States find themselves on the bank of a long and straight canal that is a mile wide. Their task is to reach Point Consensus (point C in the diagram), which is on the opposite bank of the canal, 2 miles from the point straight across from where they are standing. There is a boat they can row across the canal from point A to some point B on the other side. The boat is old and has a major leak (ha ha, pun intended), and therefore they can row it at a speed of only 2 miles/h. The opposite bank of the canal is a steep cliff the justices cannot climb. But the canal is shallow there and they can wade along the bank from point B to point C at 2.5miles/h. Find the optimal place of point B so the justices reach Point Consensus in the least amount of time by rowing across the canal and then wading along the bank. What is the minimal time for them to complete the trip? How do you know that what you found is the absolute minimum?



First, notice that this is the same problem as Example 4 in Section 3.5 with a slightly different story.

It is quite obvious that the optimal place for B should be somewhere between A' and C because it would not make it faster for the justices to land to the left of A' or to the right of C. Let x be the distance from A' to B in miles. Then we know $0 \leq x \leq 2$. By the Pythagorean Theorem, $AB = \sqrt{1+x^2}$. So it will take $\sqrt{1+x^2}/2$ hours to row from A to B. Then it will take $(2-x)/2.5$ hours to wade the remaining distance from B to C. So the total trip takes

$$f(x) = \frac{\sqrt{1+x^2}}{2} + \frac{2-x}{2.5} = \frac{\sqrt{1+x^2}}{2} + \frac{4}{5} - \frac{2}{5}x$$

hours, depending on x. The derivative

$$f'(x) = \frac{1}{2}\,\frac{1}{2}(1+x^2)^{-1/2}(2x) - \frac{2}{5}$$

for every $x \in \mathbb{R}$. So f is differentiable at every $x \in \mathbb{R}$. Hence f is continuous at every $x \in \mathbb{R}$. So we are looking for the absolute minimum of a contiunuous function over a closed interval. We can use the Closed Interval Method. Since f is differentiable, the only critical point is where

$$\begin{aligned}
0 &= f'(x) \\
&= \frac{x}{2\sqrt{1+x^2}} - \frac{2}{5} \\
\frac{x}{2\sqrt{1+x^2}} &= \frac{2}{5} \\
5x &= 4\sqrt{1+x^2} \\
25x^2 &= 16(1+x^2) \\
25x^2 &= 16 + 16x^2 \\
9x^2 &= 16 \\
x^2 &= \frac{16}{9} \\
x &= \pm\frac{4}{3}
\end{aligned}$$

Of these, only $4/3$ is in $[0, 2]$. Comparing

$$\begin{aligned}
f(0) &= \frac{\sqrt{1+0^2}}{2} + \frac{2-0}{2.5} = \frac{1}{2} + \frac{4}{5} = 1.3 \\
f\left(\frac{4}{3}\right) &= \frac{\sqrt{1+(4/3)^2}}{2} + \frac{2-4/3}{2.5} = \frac{\sqrt{25/9}}{2} + \frac{2/3}{5/2} = \frac{5}{6} + \frac{4}{15} = \frac{33}{30} = 1.1 \\
f(2) &= \frac{\sqrt{1+2^2}}{2} + \frac{2-2}{2.5} = \frac{\sqrt{5}}{2} \approx 1.12,
\end{aligned}$$

we can tell the absolute minimum of f is at 1.1 at $x = 4/3$. We know this is the absolute minimum because the absolute minimum of a continuous function on a closed interval must be either at one of the endpoints or must be a local minimum and hence a critical point, and we compared the value of f at the endpoints and at every critical point between them.

So the justices should row their boat to point B that is $4/3$ of a mile from point A' toward point C. The least amount of time for them to get to point C is 1.1 hours.

6. (a) (6 pts) Find the approximate area under the graph of $f(x) = x^3$ on the interval $[1, 3]$ using a Riemann sum with a uniform partition with four subintervals and the sample points at the midpoint of each subinterval.

The four subintervals are

$$\left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right].$$

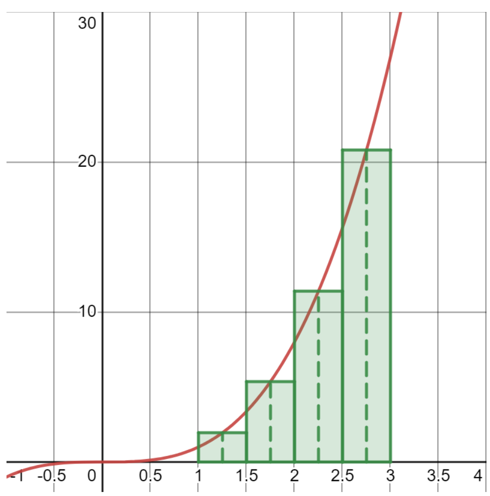
Each has width 1/2. The midpoints of these intervals are

$$\begin{aligned}
x_1^* &= 1 + \frac{1}{4} = \frac{5}{4} \\
x_2^* &= \frac{3}{2} + \frac{1}{4} = \frac{7}{4} \\
x_3^* &= 2 + \frac{1}{4} = \frac{9}{4} \\
x_4^* &= \frac{5}{2} + \frac{1}{4} = \frac{11}{4}
\end{aligned}$$

So

$$\begin{aligned}
M_4 &= \sum_{i=1}^{4} f(x_i^*)\frac{1}{2} \\
&= \frac{1}{2}\left[\underbrace{\left(\frac{5}{4}\right)^3}_{f(x_1^*)} + \underbrace{\left(\frac{7}{4}\right)^3}_{f(x_2^*)} + \underbrace{\left(\frac{9}{4}\right)^3}_{f(x_3^*)} + \underbrace{\left(\frac{11}{4}\right)^3}_{f(x_4^*)}\right] \\
&= \frac{1}{2}\,\frac{125 + 343 + 729 + 1331}{64} \\
&= \frac{79}{4} \\
&= 19.75
\end{aligned}$$

(b) (4 pts) Illustrate your calculation by drawing the rectangles whose areas correspond to the terms of your Riemann sum in the diagram below.



7. **Extra credit problem.** (5 pts each) Let m and n be a positive integers with no nontrivial common factor (that is m/n is in lowest terms). If n is odd, then let x be any real number; if n is even, then let x be any nonnegative real number. By the power rule,

$$\frac{d}{dx}x^{m/n} = \frac{m}{n}x^{m/n-1}.$$

But we never proved in class that the power rule works for powers with fractional exponents. That is what you will do in this exercise. First, let $f(x) = \sqrt[n]{x}$, and notice that

$$[f(x)]^n = \sqrt[n]{x}^n = x.$$

(a) Differentiate both sides of the equation

$$x = [f(x)]^n$$

and solve the resulting equation for $f'(x)$.

First,

$$\frac{d}{dx}x = 1.$$

Now,

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$

by the chain rule. Hence

$$1 = n[f(x)]^{n-1}f'(x) \implies f'(x) = \frac{1}{n[f(x)]^{n-1}}.$$

(b) Substitute $f(x) = \sqrt[n]{x}$ back into the expression you got in part (a) to find $f'(x)$ only in terms of x. Did you get the expected result?

$$f'(x) = \frac{1}{n[f(x)]^{n-1}} = \frac{1}{n[\sqrt[n]{x}]^{n-1}} = \frac{1}{n}\frac{1}{x^{\frac{n-1}{n}}} = \frac{1}{n}\frac{1}{x^{1-\frac{1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

This is exactly the result we would expect by using the power rule.

(c) Now let m and n be positive integers. Use $\frac{d}{dx}x^m = mx^{m-1}$ along with your result from part (b) and the chain rule to find

$$\frac{d}{dx}x^{m/n} = \frac{d}{dx}\big(\sqrt[n]{x}\big)^m.$$

Compare your result with what the power rule says $\frac{d}{dx}x^{m/n}$ should be.

By the chain rule,

$$\frac{d}{dx}x^{m/n} = \frac{d}{dx}\big(\sqrt[n]{x}\big)^m = m\big(\sqrt[n]{x}\big)^{m-1}\frac{d}{dx}\sqrt[n]{x}.$$

We know from part (b) that

$$\frac{d}{dx}\sqrt[n]{x} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

So

$$\frac{d}{dx}x^{m/n} = m\big(\sqrt[n]{x}\big)^{m-1}\frac{1}{n}x^{\frac{1}{n}-1} = \frac{m}{n}x^{\frac{m-1}{n}}x^{\frac{1}{n}-1} = \frac{m}{n}x^{\frac{m}{n}-\frac{1}{n}+\frac{1}{n}-1} = \frac{m}{n}x^{\frac{m}{n}-1}.$$

This is exactly what the power rule also says $\frac{d}{dx}x^{m/n}$ should be.