MCS 122 Exam 1 Solutions

1. (10 pts) Use the limit of a right-hand Riemann sum with a uniform partition to evaluate the integral
$$\int_0^4 x^2 - 3x\,dx.$$

Some of the following summation formulas may be helpful:
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Let $f(x) = x^2 - 3x$. If we divide the interval $[0,4]$ into n uniform subintervals, each of those will have width $4/n$. The subintervals are
$$\left[0, \frac{4}{n}\right], \left[\frac{4}{n}, 2\,\frac{4}{n}\right], \left[2\,\frac{4}{n}, 3\,\frac{4}{n}\right], \dots, \left[(n-1)\,\frac{4}{n}, 4\right]$$
Since we are using the right endpoints of these intervals, the sample points are $x_i^* = (4i)/n$. Hence the Riemann sum is
$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i^*)\frac{4}{n} \\
&= \sum_{i=1}^{n} \left(\left(\frac{4i}{n}\right)^2 - 3\,\frac{4i}{n} \right) \frac{4}{n} \\
&= \sum_{i=1}^{n} \left(\frac{16}{n^2} i^2 - \frac{12}{n} i \right) \frac{4}{n} \\
&= \frac{64}{n^3} \sum_{i=1}^{n} i^2 - \frac{48}{n^2} \sum_{i=1}^{n} i \\
&= \frac{64}{n^3}\,\frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2}\,\frac{n(n+1)}{2} \\
&= \frac{32}{3}\,\frac{2n^2+3n+1}{n^2} - 24\,\frac{n+1}{n} \\
&= \frac{32}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 24\left(1 + \frac{1}{n}\right)
\end{aligned}$$
We will now take the limit as $n \to \infty$:
$$\lim_{n\to\infty} \left[\frac{32}{3} \left(2 + \underbrace{\frac{3}{n}}_{\to 0} + \underbrace{\frac{1}{n^2}}_{\to 0} \right) - 24 \left(1 + \underbrace{\frac{1}{n}}_{\to 0} \right) \right] = \frac{64}{3} - 24 = -\frac{8}{3}.$$

2. (10 pts) Let f be a function that is twice differentiable and has an inverse g. Show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

By Theorem 7 in Section 5.1,

$$g'(x) = \frac{1}{f'(g(x))}.$$

We can differentiate this again to get

$$\begin{aligned} g''(x) &= \frac{d}{dx}[f'(g(x))]^{-1} \\ &= (-1)[f'(g(x))]^{-2} f''(g(x)) g'(x) \\ &= -\frac{f''(g(x))g'(x)}{[f'(g(x))]^2}. \end{aligned}$$

We can now substitute $g'(x) = \frac{1}{f'(g(x))}$ into this:

$$g''(x) = -\frac{f''(g(x))\frac{1}{f'(g(x))}}{[f'(g(x))]^2} = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

This is what we wanted to show.

3. (5 pts each) Find the following derivatives.

(a) $\displaystyle \frac{d}{dx}\int_1^{2^x} \frac{1}{t^3+1}\,dt$

Let

$$f(x) = \int_1^x \frac{1}{t^3+1}\,dt.$$

By the Fundamental Theorem of Calculus,

$$f'(x) = \frac{1}{x^3+1}.$$

Hence

$$\frac{d}{dx}\int_1^{2^x} \frac{1}{t^3+1}\,dt = \frac{d}{dx}f(2^x) = f'(2^x)\frac{d}{dx}2^x = \frac{1}{\left(2^x\right)^3+1}\,2^x\ln(x) = \frac{2^x\ln(x)}{2^{3x}+1}.$$

(b) $\displaystyle \frac{d}{dx}(x^2+3)^{\ln(x)}$

We can either use logarithmic differentiation, or we can rewrite

$$(x^2+3)^{\ln(x)} = \left(e^{\ln(x^2+3)}\right)^{\ln(x)} = e^{\ln(x^2+3)\ln(x)}.$$

Following the second approach,

$$\begin{aligned} \frac{d}{dx}(x^2+3)^{\ln(x)} &= \frac{d}{dx}e^{\ln(x^2+3)\ln(x)} \\ &= e^{\ln(x^2+3)\ln(x)}\,\frac{d}{dx}\big(\ln(x^2+3)\ln(x)\big) \\ &= (x^2+3)^{\ln(x)}\left(\frac{1}{x^2+3}\,2x\,\ln(x) + (x^2+3)\,\frac{1}{x}\right) \\ &= (x^2+3)^{\ln(x)}\left(\frac{2x\ln(x)}{x^2+3} + \frac{\ln(x^2+3)}{x}\right) \end{aligned}$$

To use logarithmic differentiation, let $y = (x^2+3)^{\ln(x)}$. Then

$$\ln(y) = \ln\left((x^2+3)^{\ln(x)}\right) = \ln(x)\ln(x^2+3)$$

and

$$\begin{aligned}
\frac{d}{dx}\ln(y) &= \frac{d}{dx}\big(\ln(x^2+3)\ln(x)\big) \\
\frac{1}{y}y' &= \frac{1}{x^2+3}\,2x\,\ln(x) + (x^2+3)\,\frac{1}{x} \\
&= \frac{2x\ln(x)}{x^2+3} + \frac{\ln(x^2+3)}{x}
\end{aligned}$$

Finally,

$$y' = y\left(\frac{2x\ln(x)}{x^2+3} + \frac{\ln(x^2+3)}{x}\right) = (x^2+3)^{\ln(x)}\left(\frac{2x\ln(x)}{x^2+3} + \frac{\ln(x^2+3)}{x}\right).$$

4. (10 pts) We defined the natural log function $\ln : \mathbb{R}^+ \to \mathbb{R}$ by

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

Prove that for all $x, y \in \mathbb{R}^+$,

$$\ln(xy) = \ln(x) + \ln(y).$$

Hint: We started by differentiating both sides.

We will differentiate each side with respect to x, treating y as a constant (or as a variable that is independent of x, if you prefer to think about it that way):

$$\frac{d}{dx}\ln(xy) = \frac{1}{xy}\frac{d}{dx}(xy) = \frac{1}{xy}\,y = \frac{1}{x}$$

and

$$\frac{d}{dx}\big(\ln(x) + \ln(y)\big) = \frac{1}{x} + 0 = \frac{1}{x}.$$

Since $\ln(xy)$ and $\ln(x)+\ln(y)$ have the same derivative on the interval $(0,\infty)$, they can only differ by a constant c according to Corollary 7 in Section 3.2. Hence

$$\ln(xy) = \ln(x) + \ln(y) + c.$$

To find the value of c, we will substitute $x = 1$ and use that

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0.$$

If $x = 1$, then

$$\ln(1\cdot y) = \ln(1) + \ln(y) + c = \ln(y) + c.$$

This shows $c = \ln(y) - \ln(y) = 0$. Therefore

$$\ln(xy) = \ln(x) + \ln(y)$$

for all $x, y \in \mathbb{R}^+$,

5. (10 pts) **Extra credit problem.** When we defined

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

the choice of 1 for the lower bound may have seemed like an arbitrary choice. In fact, there is good reason to choose 1. Suppose we defined

$$\ln(x) = \int_a^x \frac{1}{t}\,dt$$

where a is some positive real number. Show that for

$$\ln(xy) = \ln(x) + \ln(y)$$

to be true for all $x, y \in \mathbb{R}^+$, we must have $a = 1$.

Suppose

$$\ln(xy) = \ln(x) + \ln(y)$$

is true for all $x, y \in \mathbb{R}^+$. Then for $x = y = 1$, we get

$$\ln(1) = \ln(1) + \ln(1) \implies \ln(1) = 0.$$

Hence

$$0 = \int_a^1 \frac{1}{t}\,dt.$$

We know $f(t) = 1/t$ is a decreasing function on $(0, \infty)$ since $f'(t) = -1/t^2 < 0$. Hence if $0 < a < 1$, then $1/t \geq 1$ for all $t \in [a, 1]$, and so

$$\int_a^1 \frac{1}{t}\,dt \geq 1 \cdot (1 - a) > 0.$$

This contradicts

$$0 = \int_a^1 \frac{1}{t}\,dt,$$

and so $a < 1$ cannot be true.

If $a > 1$, then $1/t > 1/a$ for all $t \in [1, a]$, and so

$$\int_1^a \frac{1}{t}\,dt \geq \frac{1}{a}\,(a - 1) > 0.$$

It follows that

$$\int_a^1 \frac{1}{t}\,dt = -\int_1^a \frac{1}{t}\,dt$$

cannot be 0. We have reached a contradiction again, and so $a > 1$ cannot be true either.

Hence a must be 1 for $\ln(xy) = \ln(x) + \ln(y)$ to be true for all $x, y \in \mathbb{R}^+$.