MCS 122 Exam 2 Solutions

1. (10 pts) Let a and b be real numbers whose signs are opposite. That is one of the two numbers is positive while the other is negative. Show that there exist real numbers α and β such that
$$ae^x + be^{-x} = \alpha \sinh(x + \beta)$$
for all $x \in R$.

We will start by expanding $\alpha \sinh(x + \beta)$ in terms of exponential functions:
$$\alpha \sinh(x + \beta) = \alpha \frac{e^{x+\beta} - e^{-x-\beta}}{2} = \frac{\alpha e^\beta}{2} e^x - \frac{\alpha e^{-\beta}}{2} e^{-x}.$$
Comparing the coefficients of e^x and e^{-x} with those in $ae^x + be^{-x}$ tells us that we want
$$a = \frac{\alpha e^\beta}{2}$$
$$b = -\frac{\alpha e^{-\beta}}{2}$$
We will now solve these equations for α and β. First, multiply the two equations together:
$$ab = -\frac{\alpha e^\beta}{2} b \frac{\alpha e^{-\beta}}{2} = -\frac{\alpha^2}{4}.$$
Hence
$$\alpha^2 = -4ab \implies \alpha = \pm\sqrt{-4ab} = \pm 2\sqrt{-ab}.$$
Note that a and b have opposite signs, so $-ab$ cannot be negative. Since e^β is always positive, a and α must have the same sign. This tells us we must choose $\alpha = 2\sqrt{-ab}$ if a is positive and $\alpha = -2\sqrt{-ab}$ if a is negative. Now that we know the value of α, we can substitute it back into the first equation and solve for β:
$$a = \frac{\alpha e^\beta}{2} \implies e^\beta = \frac{2a}{\alpha} \implies \beta = \ln\left(\frac{2a}{\alpha}\right).$$
Note that a and α have the same sign, and so the input to the natural log is always positive here. We will now show that with these values of α and β, it is indeed true that $ae^x + be^{-x} = \alpha \sinh(x + \beta)$:
$$\begin{aligned}
\alpha \sinh(x + \beta) &= \frac{\alpha e^\beta}{2} e^x - \frac{\alpha e^{-\beta}}{2} e^{-x} \\
&= \frac{\alpha e^{\ln\left(\frac{2a}{\alpha}\right)}}{2} e^x - \frac{\alpha e^{-\ln\left(\frac{2a}{\alpha}\right)}}{2} e^{-x} \\
&= \frac{\alpha\,\frac{2a}{\alpha}}{2} e^x - \frac{\alpha\,\frac{\alpha}{2a}}{2} e^{-x} \\
&= ae^x - \frac{\alpha^2}{4a} e^{-x} \\
&= ae^x - \frac{-4ab}{4a} e^{-x} && \text{since } \alpha^2 = -4ab \\
&= ae^x + be^{-x}.
\end{aligned}$$

2. (10 pts) Use l'Hôpital's Rule to evaluate
$$\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}.$$

Be sure to fully justify your argument, including why you can use l'Hôpital's Rule to evaluate this limit.

First we will rewrite $(e^x + x)^{1/x}$ as

$$(e^x + x)^{1/x} = e^{\ln[(e^x+x)^{1/x}]} = e^{\frac{\ln(e^x+x)}{x}}.$$

Let us now focus on the exponent. Since $x \to \infty$, we may assume that x is positive. So $e^x + x \geq e^x$, and since the natural log is an increasing function, $\ln(e^x + x) \geq \ln(e^x) = x$. Since x gets arbirarily large as $x \to \infty$, so does $\ln(e^x + x)$. And of course, the denominator x goes to ∞. Hence we can use l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln(e^x + x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{e^x+x}(e^x + 1)}{1} = \lim_{x\to\infty} \frac{e^x + 1}{e^x + x}.$$

We have already observed that $\lim_{x\to\infty}(e^x+x) = \infty$. For much the same reason, $\lim_{x\to\infty}(e^x+1) = \infty$ too. So l'Hôpital's Rule applies once again:

$$\lim_{x\to\infty} \frac{e^x + 1}{e^x + x} = \lim_{x\to\infty} \frac{e^x}{e^x + 1}.$$

This is still in ∞/∞ form, as we have already shown, and hence we can use l'Hôpital's Rule omne more time:

$$\lim_{x\to\infty} \frac{e^x}{e^x + 1} = \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} 1 = 1.$$

Combining the above yields

$$\lim_{x\to\infty} \frac{\ln(e^x + x)}{x} = 1.$$

Finally,

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\frac{\ln(e^x+x)}{x}} = e^{\lim_{x\to\infty} \frac{\ln(e^x+x)}{x}} = e^1 = e$$

because the function $f(x) = e^x$ is continuous.

3. (10 pts) When the town bank is reported robbed overnight, investigators arrive at the scene at 8:30 AM, and immediately notice that the shattered lock of the vault is very cold. In fact, they measure its temperature at -191 °C. They quickly realize that the robber broke the vault's lock by pouring liquid helium in the lock and hitting the rigid metal with a 30-pound sledgehammer found at the scene. At 9:00 AM, the temperature of the lock is measured again and it is -156 °C. The thermostat in the bank is set at 24 °C. The boiling point of helium is -271 °C.
The police's primary suspect is Jay Walker, former locksmith, small-time crook, and long-suspected criminal mastermind. Jay is currently in jail for carjacking an icecream truck near the bank. He was arrested at 7:15 AM driving that truck with a large, heavy-duty thermos bottle on the passenger seat filled with pistachio icecream. So Jay's has alibi after 7:15 AM is rock-solid. Could he have committed the robbery?

Let $T(t)$ be the temperature of the lock at time t, where t is measured in minutes relative to 8:30 AM. That is $T(0) = -191$ °C. Let $y(t) = T(t) - 24$ be temperature difference between the lock and its environment. By Newton's Law of Cooling, $y = k\frac{dy}{dt}$ for some constant k. We know that the solution of such a differential equation is a function $y(t) = y_0 e^{kt}$. Here is what we know:

$$\begin{aligned} y(0) = (-191-24)e^0 = -215 &\implies y_0 = -215 \\ y(30) = y_0 e^{30k} &\implies -180 = -215e^{30k} \\ &\implies e^{30k} = \frac{180}{215} \\ &\implies e^k = \sqrt[30]{\frac{36}{43}} \\ &\implies k = \frac{\ln(36/43)}{30}. \end{aligned}$$

While we calculated a value for k, you will see that it is enough to know e^k. Now, we want to find t such that $T(t) = -271$, or $y(t) = -271 - 24 = -295$. So

$$-295 = -215e^{kt} = -215\left(e^k\right)^t = -215\sqrt[30]{\frac{36}{43}}^{\,t} \implies \frac{295}{215} = \left(\frac{36}{43}\right)^{t/30}.$$

Simplifying $295/215$ and taking natural log of both sides gives

$$\ln\left(\frac{59}{43}\right) = \frac{t}{30}\ln\left(\frac{36}{43}\right) \implies t = 30\frac{\ln\left(\frac{59}{43}\right)}{\ln\left(\frac{36}{43}\right)} = 30\frac{\ln(59)-\ln(43)}{\ln(36)-\ln(43)} \approx -53.4.$$

So the lock was broken about 53 minutes before 8:30 AM, that is at about 7:37 AM. Jay was in police custody by then, hence he could not have been the one to break into the vault.

4. (10 pts) Let the function $f : [-\pi/2, \pi/2] \to \mathbb{R}$ be $f(x) = \tan(x)$ and note that f is both one-to-one and onto. Define the arctan function to be the inverse of f. Prove that

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

As usual, we can differentiate both sides of the equation $x = \tan(\arctan(x))$:

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx}\tan(\arctan(x)) \\ 1 &= \sec^2(\arctan(x))\frac{d}{dx}\arctan(x) \\ \frac{d}{dx}\arctan(x) &= \frac{1}{\sec^2(\arctan(x))} \end{aligned}$$

To simplify $\sec^2(\arctan(x)$, we can use the trig identity $\sec^2(x) = 1 + \tan^2(x)$:

$$\frac{d}{dx}\arctan(x) = \frac{1}{\sec^2(\arctan(x)} = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}.$$

5. **Extra credit problem.** Let f and g be functions of real numbers and $a \in \mathbb{R}$ such that f and g are differentiable and $g'(x)$ at every x in some neighborhood $(a-\delta, a+\delta)$ except possibly at a. Suppose $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists. We proved in class that if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0,$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

That is l'Hôpital's Rule holds in the 0/0 indeterminate case.

The goal of this exercise is to prove that if

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty,$$

then it is also true that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

in the somewhat special case when $\lim_{x\to a} \frac{f'(x)}{g'(x)} \neq 0$.

(a) (4 pts) Suppose f and g satisfy the above hypotheses, including

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty.$$

Since $f(x)$ must be either large or very negative for x near a, we may assume that the δ above is sufficiently small that $f(x) \neq 0$ for all x in the neighborhood $(a-\delta, a+\delta)$. Let $F(x) = 1/f(x)$ and $G(x) = 1/g(x)$. It should be clear that

$$\lim_{x\to a} f(x) = \pm\infty \implies \lim_{x\to a} F(x) = 0$$
$$\lim_{x\to a} g(x) = \pm\infty \implies \lim_{x\to a} G(x) = 0.$$

Prove that F and G satisfy the remaining conditions for l'Hôpital's Rule in the 0/0 indeterminate case.

We already know that $\lim_{x\to a} F(x) = \lim_{x\to a} G(x) = 0$. Since $G(x) = 1/g(x)$, it is clear that $G(x) \neq 0$ near a. Finally,

$$\frac{d}{dx}F(x) = \frac{d}{dx}\frac{1}{f(x)} = -\frac{f'(x)}{f^2(x)}$$
$$\frac{d}{dx}G(x) = \frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{g^2(x)}$$

by the chain rule. Since f and g are differentiable near a, the numerators exist. We have already noted that $f(x) \neq 0$ and $g(x) \neq 0$ near a, hence the denominators are not 0. Therefore both F and G are differentiable near a. So all of the conditions of l'Hôpital's Rule as satisfied.

(b) (6 pts) Suppose

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)} \neq 0.$$

Use l'Hôpital's Rule in the 0/0 indeterminate case to prove

$$\lim_{x\to a} \frac{F(x)}{G(x)} = \frac{1}{L},$$

and conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L$$

must be true.

This is actually quite tricky. I should have given you a more detailed hint on how to get started. The idea is similar to how we proved l'Hôpital's Rule in the 0/0 case, but we cannot just replace f with a function F such that

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ \pm\infty & \text{if } x = a \end{cases}$$

and g with a similar function G. So we will use $F(x) = 1/f(x)$ and $G(x) = 1/g(x)$. We have already noted in part (a) that F and G satisfy the conditions of l'Hôpital's Rule in the 0/0 case. Except we do not know if $\lim_{x\to a} \frac{F(x)}{G(x)}$ exists. To get around these two difficulties, we will look at the value of

$$\frac{f(x) - f(y)}{g(x) - g(y)}$$

when x and y are both close to a but not equal to a. We may assume without loss of generality that x is closer to a than y, otherwise we can just switch the two numbers. Suppose y is close enough to a that f and g are both differentiable and continuous on the interval $[x, y]$ (or $[y, x]$ if $y < x$, but for the sake of keeping the notation manageable, let us keep this as $[x, y]$ with the understanding that x and y may have to switched if they are in the wrong order). By Cauchy's Mean Value Theorem, there exists some $c \in (x, y)$

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}.$$

Now, note that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(x)}{g(x)} \frac{1 - \frac{f(y)}{f(x)}}{1 - \frac{g(y)}{g(x)}} = \frac{f(x)}{g(x)} \frac{1 - f(y)F(x)}{1 - g(y)G(x)}.$$

Hence

$$\frac{f(x)}{g(x)} \frac{1 - f(y)F(x)}{1 - g(y)G(x)} = \frac{f'(c)}{g'(c)}$$

and it follows that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \frac{1 - g(y)G(x)}{1 - f(y)F(x)}.$$

First, notice we can make $f'(c)/g'(c)$ be as close to L as we want by choosing y–and hence x and c–to be sufficiently close to a. Now, keeping y fixed and letting x approach a, we see

$$\lim_{x\to a} \frac{1 - g(y)G(x)}{1 - f(y)F(x)} = \frac{\lim_{x\to a} \big(1 - g(y)G(x)\big)}{\lim_{x\to a} \big(1 - f(y)F(x)\big)} = \frac{1 - g(y) \lim_{x\to a} G(x)}{1 - f(y) \lim_{x\to a} F(x)} = 1$$

since $G(x)$ and $F(x)$ both approach 0 as $x \to a$. This means that by choosing x and y close enough to a, we can make $f'(c)/g'(c)$ as close as we want to L and $\frac{1-g(y)G(x)}{1-f(y)F(x)}$ as close to 1 as we want. By doing both, we can make

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \frac{1 - g(y)G(x)}{1 - f(y)F(x)}.$$

as close to L as we want. Hence

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

You can make this argument more rigorous by using the $\delta - \epsilon$ definition of the limit. I will leave the details to you.