MCS 122 Final Exam Solutions

1. (10 pts) The function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = x^3 + 3\sin(x) + 2\cos(x)$$
is one-to-one. Find $\left(f^{-1}\right)'(2)$.

First, we will need to know $f^{-1}(2)$ to calculate the derivative of f^{-1} at 2. Finding a formula for f^{-1} in general is hopeless, but it is easy enough to see that
$$f(0) = 0^3 + 2\sin(0) + 2\cos(0) = 2,$$
and so $f^{-1}(2) = 0$. We can now use the usual formula
$$\left(f^{-1}\right)'(2) = \frac{1}{f'\left(f^{-1}(2)\right)} = \frac{1}{f'(0)}.$$
Since $f'(x) = 3x^2 + 3\cos(x) - 2\sin(x)$, we have
$$f'(0) = 3 \cdot 0^2 + 3\cos(0) - 2\sin(0) = 3.$$
Therefore $\left(f^{-1}\right)'(2) = 1/3$.

2. (a) (7 pts) Prove the reduction formula
$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.$$

We start with
$$\int \cos^n(x)\,dx = \int \cos^{n-1}(x)\ \cos(x)dx.$$
Let $u = \cos^{n-1}(x)$ and $dv = \cos(x)dx$. Then $du = (n-1)\cos^{n-2}(x)(-\sin(x))dx$ and $v = \sin(x)$. Hence
$$\begin{aligned}\int \cos^n(x)\,dx &= \cos^{n-1}(x)\sin(x) - \int (n-1)\cos^{n-2}(x)(-\sin^2(x))\,dx\\ &= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\,dx\\ &= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)(1-\cos^2(x))\,dx\\ &= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x) - \cos^n(x)\,dx\\ &= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^n(x)\,dx\end{aligned}$$
Let us add $(n-1)\int \cos^n(x)\,dx$ to both sides:
$$n\int \cos^n(x)\,dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx.$$
Dividing by n, we find
$$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.$$

(b) (3 pts) Use the formula in part (a) to evaluate $\int \cos^2(x)\,dx$.

We just need to use the formula with $n = 2$:

$$\int \cos^2(x)\,dx = \frac{\cos(x)\sin(x)}{2} + \frac{1}{2}\int dx = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c.$$

3. (10 pts) Suppose the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent. Let a and b be real numbers. Show that

$$\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx.$$

Hint: We know the property $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ if a, b, and c are real numbers.

First, note that since $\int_{-\infty}^{\infty} f(x)\,dx$ is convergent, the following four improper integrals must all be convergent:

$$\int_{-\infty}^{a} f(x)\,dx,\ \int_{a}^{\infty} f(x)\,dx,\ \int_{-\infty}^{b} f(x)\,dx,\ \int_{b}^{\infty} f(x)\,dx.$$

To make the rest of the argument easier to visualize, let us suppose $a \leq b$. If $a > b$, just switch a and b. We do not actually need to know $a \leq b$ for what we are about to do, but it is easier to visualize what is going on in terms of areas under graphs if we assume $a \leq b$. First,

$$\begin{aligned}
\int_{-\infty}^{b} f(x)\,dx &= \lim_{t \to -\infty} \int_{t}^{b} f(x)\,dx \\
&= \lim_{t \to -\infty} \left(\int_{t}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx \right) \\
&= \lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx + \lim_{t \to -\infty} \int_{a}^{b} f(x)\,dx.
\end{aligned}$$

Note that $\int_a^b f(x)\,dx$ does not depend on t at all, so its limit as $t \to -\infty$ is just $\int_a^b f(x)\,dx$. Hence

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{b} f(x)\,dx.$$

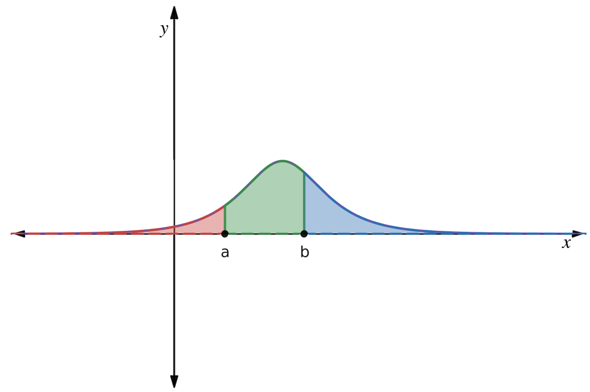
By an analogous argument

$$\int_{a}^{\infty} f(x)\,dx = \int_{a}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx.$$

Therefore

$$\begin{aligned}
\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx &= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx.
\end{aligned}$$

This is easy to visualize in terms of areas:



4. (10 pts) Use the method of partial fractions to evaluate the indefinite integral

$$\int \frac{5x^4 - 2x^3 + 40x^2 - 7x + 80}{x(x^2+4)^2}\,dx.$$

We will let

$$\begin{aligned}\frac{5x^4 - 2x^3 + 40x^2 - 7x + 80}{x(x^2+4)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}\\ &= \frac{A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x}{x(x^2+4)^2}.\end{aligned}$$

Since the denominators are the same, the numerators must be equal:

$$5x^4 - 2x^3 + 40x^2 - 7x + 80 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x.$$

If $x = 0$, we get $80 = 16A$, and so $A = 5$. To find the values of the remaining parameters, we can multiply out on the right-hand side:

$$\begin{aligned}5x^4 - 2x^3 + 40x^2 - 7x + 80 &= 5(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x\\ &= 5x^4 + 40x^2 + 80 + Bx^4 + 4Bx^2 + Cx^3 + 4Cx + Dx^2 + Ex\\ &= (5+B)x^4 + Cx^3 + (40+4B+D)x^2 + (4C+E)x + 80.\end{aligned}$$

Setting corresponding coefficients equal to each other gives

$$\begin{aligned}5 + B = 5 &\implies B = 0\\ C &= -2\\ 40 + 4B + D = 40 &\implies D = 40 - (40+4B) = 0\\ 4C + E = -7 &\implies E = -7 - 4C = 1.\end{aligned}$$

Let us check that our calculation so far is correct, so we do not spend time and effort integrating the wrong function:

$$\begin{aligned}\frac{5}{x} - \frac{2}{x^2+4} + \frac{1}{(x^2+4)^2} &= \frac{5(x^2+4)^2 - 2x(x^2+4) + x}{x(x^2+4)^2}\\ &= \frac{5x^4 + 40x^2 + 80 - 2x^3 - 8x + x}{x(x^2+4)^2}\\ &= \frac{5x^4 - 2x^3 + 40x^2 - 7x + 80}{x(x^2+4)^2}.\end{aligned}$$

We can now do the integration:

$$\int \frac{5x^4 - 2x^3 + 40x^2 - 7x + 80}{x(x^2+4)^2}\,dx = \int \frac{5}{x} - \frac{2}{x^2+4} + \frac{1}{(x^2+4)^2}\,dx$$
$$= 5\ln|x| - 2\int \frac{1}{x^2+4}\,dx + \int \frac{1}{(x^2+4)^2}\,dx.$$

To evaluate the second integral, we can substitute $x = 2u$. So $dx = 2du$ and $u = x/2$. Then

$$\int \frac{1}{x^2+4}\,dx = \int \frac{2}{4u^2+4}\,du = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\arctan(u) + c = \frac{\arctan(x/2)}{2} + c.$$

For the third integral, we will substitute $x = 2\tan(u)$. So $dx = 2\sec^2(u)du$. Hence

$$\begin{aligned}
\int \frac{1}{(x^2+4)^2}\,dx &= \int \frac{2\sec^2(u)}{(4\tan^2(u)+4)^2}\,du \\
&= \frac{2}{16}\int \frac{\sec^2(u)}{(\tan^2(u)+1)^2}\,du \\
&= \frac{1}{8}\int \frac{\sec^2(u)}{\left(\sec^2(u)\right)^2}\,du \\
&= \frac{1}{8}\int \frac{1}{\sec^2(u)}\,du \\
&= \frac{1}{8}\int \cos^2(u)\,du \\
&= \frac{1}{8}\left(\frac{\cos(u)\sin(u)}{2} + \frac{u}{2}\right) + c.
\end{aligned}$$

by using the result of problem 2b. We can now subsitute x back into this result. Since $x = 2\tan(u)$, we have $u = \arctan(x/2)$. Here is a way to express $\sin(u)\cos(u)$ in terms of x in simple terms:

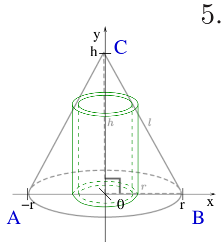
$$\begin{aligned}
\sin(u)\cos(u) &= \frac{\sin(u)}{\cos(u)}\cos^2(u) \\
&= \frac{\tan(u)}{\sec^2(u)} \\
&= \frac{\tan(u)}{\tan^2(u)+1} \\
&= \frac{\frac{x}{2}}{\left(\frac{x}{2}\right)^2+1} \\
&= \frac{2x}{x^2+4}.
\end{aligned}$$

Hence

$$\int \frac{1}{(x^2+4)^2}\,dx = \frac{1}{8}\left(\frac{x}{x^2+4} + \frac{\arctan(x/2)}{2}\right) + c = \frac{1}{8}\frac{x}{x^2+4} + \frac{\arctan(x/2)}{16} + c.$$

Putting the pieces together:

$$\int \frac{5x^4 - 2x^3 + 40x^2 - 7x + 80}{x(x^2+4)^2}\,dx = 5\ln|x| - \arctan\left(\frac{x}{2}\right) + \frac{1}{8}\frac{x}{x^2+4} + \frac{\arctan(x/2)}{16} + c$$
$$= 5\ln|x| - \frac{15}{16}\arctan\left(\frac{x}{2}\right) + \frac{1}{8}\frac{x}{x^2+4} + c.$$



5. (10 pts) A right circular cone is a cone whose base is a circle and the line segment (called the axis) that connects the vertex of the cone and the center of the base is perpendicular to the base. Use cylindrical shells to set up an integral for the volume of a right circular cone of radius r and h, and find the volume by evaluating the integral.

Let the x-axis lie on one of the radii of the circular base and the y-axis along the axis of the cone. So the origin is at the center of the base. The cross-section of such a cone in the xy-plane is an isosceles triangle with its three vertices at $A = (-r, 0)$, $B = (r, 0)$, and $C = (0, h)$. So line BC has slope $-h/r$ and y-intercept h. Hence its equation is $y = (-h/r)x + h$. To find the volume, we will use cylindrical shells whose axis of rotation is the y-axis. Let x be the inner radius of such a shell and Δx its thickness. The height of the shell is $y = h - (h/r)x$. Using the usual formula, the volume is

$$\begin{aligned}\int_0^r 2\pi x\left(h - \frac{h}{r}x^2\right)\,dx &= 2\pi\int_0^r hx - \frac{h}{r}x^2\,dx \\ &= 2\pi h\left[\frac{x^2}{2} - \frac{x^3}{3r}\right]_0^r \\ &= 2\pi h\left(\frac{r^2}{2} - \frac{r^3}{3r}\right) \\ &= 2\pi hr^2\left(\frac{1}{2} - \frac{1}{3}\right) \\ &= \frac{\pi hr^2}{3}.\end{aligned}$$

6. (10 pts) Find the limit of the sequence $a_n = \sqrt[n]{n^2+3}$.

Hint: Consider the limit of the corresponding function $f(x) = \sqrt[x]{x^2+3}$ as $x \to \infty$.

By Theorem 3 in Section 8.1,

$$\lim_{n\to\infty} \sqrt[n]{n^2+3} = \lim_{x\to\infty} \sqrt[x]{x^2+3}$$

if the limit on the right-hand side exists. We can use l'Hôpital's Rule to evaluate it.

$$\begin{aligned}\lim_{x\to\infty} \sqrt[x]{x^2+3} &= \lim_{x\to\infty} e^{\ln(\sqrt[x]{x^2+3})} \\ &= \lim_{x\to\infty} e^{\frac{\ln(x^2+3)}{x}}\end{aligned}$$

Let us focus our attention on the limit of the exponent. As $x \to \infty$, it is clear that $x^2 + 3 \to \infty$, and hence $\ln(x^2+3) \to \infty$. So the limit of the indeterminate form ∞/∞, and we can use l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln(x^2+3)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x^2+3}2x}{1} = \lim_{x\to\infty} \frac{2x}{x^2+3}.$$

It is clear that this is again of the indeterminate form ∞/∞, so we could use l'Hôpital's rule again, or we can use the more elementary method of dividing both the numerator and the denominator by x^2 and using the limit laws:

$$\lim_{x\to\infty} \frac{2x}{x^2+3} = \lim_{x\to\infty} \frac{\frac{2x}{x^2}}{\frac{x^2+3}{x^2}} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1+\frac{3}{x^2}} = \frac{2\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 1 + 3\lim_{x\to\infty}\frac{1}{x^2}} = \frac{0}{1} = 0.$$

Since the function $g(x) = e^x$ is continuous at every real number, and in particular at 0,

$$\lim_{x\to\infty} e^{\ln(\sqrt[x]{x^2+3})} = e^{\lim_{x\to\infty}\ln(\sqrt[x]{x^2+3})} = e^0 = 1.$$

We can now conclude

$$\lim_{n\to\infty} \sqrt[n]{n^2+3} = \lim_{x\to\infty} \sqrt[x]{x^2+3} = 1.$$

7. (5 pts each) **Extra credit problem.** Define the sequence a_n recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{1+a_n}$ for $n \in \mathbb{Z}^+$. It is easy enough to see that this is the sequence

$$\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \ldots$$

The purpose of this exercise is to show that this sequence is convergent and its limit is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. I will break this down into a few steps for you.

Before we begin, note that ϕ is one of the two roots of the quadratic polynomial $x^2 - x - 1$. The other root is the negative number $\frac{1-\sqrt{5}}{2}$.

(a) It should be clear that a_n is positive for all $n \in \mathbb{Z}^+$. Prove that if $a_n < \phi$, then $a_{n+1} < \phi$.

Hint: Since the graph of $x^2 - x - 1$ is an upright parabola, if $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$, then $x^2 - x - 1 < 0$.

Suppose $a_n < \phi$. Since a_n is clearly positive, we know

$$\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}.$$

So a_n is between the roots of the quadratic function $f(x) = x^2 - x - 1$, whose graph is an upright parabola. Hence

$$0 > f(a_n) = a_n^2 - a_n - 1 \implies a_n^2 < a_n + 1 < \phi + 1.$$

But ϕ is one of the roots of $x^2 - x - 1$, and so

$$0 = \phi^2 - \phi - 1 \implies \phi + 1 = \phi^2.$$

Hence $a_n^2 < a_n + 1 < \phi + 1 = \phi^2$. Since the function $g(x) = \sqrt{x}$ is increasing on $[0, \infty)$, and a_n and ϕ are both positive, we get

$$a_n^2 < \phi^2 \implies \sqrt{a_n^2} < \sqrt{\phi^2} \implies a_n < \phi.$$

(b) It follows from part (a), that since $a_1 = 1 < \phi$, we also know $a_2 < \phi$, and so $a_3 < \phi$, and so on. That is $\{a_n\}$ is bounded above by ϕ. Let us show that it is also an increasing sequence. Use that $0 < a_n < \phi$ to show that $a_{n+1} > a_n$. Conclude that $\{a_n\}$ must be convergent.

In part (a), we actually showed that if $a_n < \phi$, then $a_n^2 < a_n + 1$. Notice that $a_n + 1 = a_{n+1}^2$. By part (a), We know $a_n < \phi$ is true for all $n \in \mathbb{Z}^+$. Therefore $a_n^2 < a_{n+1}^2$ is also true for all $n \in \mathbb{Z}^+$. Remember that the square root function is increasing on $[0, \infty)$, so we can take square roots of both sides:

$$a_n^2 < a_{n+1}^2 \implies \sqrt{a_n^2} < \sqrt{a_{n+1}^2} \implies a_n < a_{n+1}.$$

This is true for all $n \in \mathbb{Z}^+$, hence the sequence $\{a_n\}$ is increasing. It is also bounded above by ϕ. It follows by the Monotonic Sequence Theorem, that it must be convergent.

(c) Now that we know $\{a_n\}$ is convergent, let L be its limit. That is

$$\lim_{n\to\infty} a_n = L.$$

It should be clear that

$$\lim_{n\to\infty} a_{n+1} = L$$

too. Now use that $a_{n+1} = \sqrt{1 + a_n}$ to prove that $L = \sqrt{1 + L}$. Conclude that L must be a root of the polynomial $x^2 - x - 1$, and since L is clearly positive, $L = \phi$.

By using the various limit laws for sequences, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{1 + a_n} = \sqrt{\lim_{n\to\infty} (1 + a_n)} = \sqrt{\lim_{n\to\infty} 1 + \lim_{n\to\infty} a_n} = \sqrt{1 + L}.$$

But we also know $\lim_{n\to\infty} a_{n+1} = L$. So

$$L = \lim_{n\to\infty} a_{n+1} = \sqrt{1 + L} \implies L^2 = 1 + L.$$

That is L is a root of the polynomial $x^2 - x - 1$. As we have observed, that polynomial has two roots: a negative root and a positive root. Since L is the limit of a sequence of positive numbers, L cannot be negative. Therefore L must be the positive root of $x^2 - x - 1$. That root is the golden ratio ϕ. Hence

$$\lim_{n\to\infty} a_n = \phi.$$

Notice how neat this result is! By slightly abusing notation, we can say

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}.$$