MCS 122 Review Sheet

Here is a list of topics we have covered so far. Your review should certainly include your homework problems, as some of these will show up on the exam. You should also review the problems in your Webwork homework as the problem solving strategies you practiced while working on those may prove useful in solving problems on the exam.

The word "understand" is often used below. The definition of understanding is that you understand something when you know why it is true and can give a coherent and correct argument (proof) to convince someone else that it is true.

- The area under the graph
  - Approximating the value of a function from its derivative. The distance problem: approximating distance traveled by summing the product of velocity by time over short time intervals.
  - Approximating the area under the graph of f by rectangles. The Riemann sum, partition of an interval, sample points. Special cases of the Riemann sum: left-hand sums, right-hand sums, the midpoint rule, lower sums, upper sums. Understand how to calculate a Riemann-sum approximation using a finite number of rectangles.
  - Area under the graph defined as the limit of the Riemann sum. Calculating such a limit directly from the definition, using properties of sums and limits. Definition of the definite integral and integrable function. Integral notation, dummy variables. Understand why areas under the x-axis count as negative.
  - Theorem: A function f that is continuous or has only a finite number of jump discontinuities on a closed interval $[a, b]$ is integrable on $[a, b]$.
  - Understand how the definite integral can be found in some special cases using area formulas for familiar shapes and symmetry instead of calculating limits.
  - Properties of the definite integral:
    * $\int_a^b c\,dx = c(b-a)$
    * $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
    * $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
    * $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
    * If $0 \le f(x)$ for all $x \in [a, b]$ then $0 \le \int_a^b f(x)\,dx$
    * If $f(x) \ge g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$
    * If $m \le f(x) \le M$ for all $x \in [a, b]$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
  - The definition of the indefinite integral of f as the most general antiderivative of f. Make sure you understand the difference between the definite and the indefinite integrals: the former is the area under the graph (that is a number), the latter is the antiderivative (that is a function).
  - The Fundamental Theorem of Calculus establishes the connection between integration and differentiation: If f is a continuous function on the interval $[a, b]$ then
    1. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ for any $x \in [a, b]$.

2. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative (or indefinite integral) of f, that is $F'(x) = f(x)$.

- Understand how to use the FTC and the chain rule to differentiate integrals that have variables in their lower/upper bounds.
- Integration by substituion: understand how substitution allows you to integrate functions by doing the chain rule backwards. You should be able to do both indefinite and definite integrals this way.
- The average value of a function f over an interval. The Mean Value Theorem for Integrals and its proof.

- Inverse functions
  - Definition of the inverse of a function $f(x)$. How to find the inverse by using $y = f^{-1}(x)$ if and only if $x = f(y)$ and solving for y or by solving $f(g(x)) = x$ for $g(x)$.
  - The domain and codomain of f^{-1} and its graph as a mirror image of the graph of f.
  - One-to-one and onto functions. Definition, examples. The horizontal line test. Theorem: a function f has an inverse if and only if f is both one-to-one and onto.
  - Theorem: If f is a continuous function that has an inverse f^{-1} then f^{-1} is also continuous.
  - If f is a differentiable function that has an inverse f^{-1}, then f^{-1} is also differentiable and
  $$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$
  - We noted that a function that is increasing or decreasing is one-to-one. We have also seen some examples of how we can make a function one-to-one or onto by restricting its domain or codomain.

- The natural log function
  - Important example of using the FTC: we defined the natural log function by $\ln(x) = \int_1^x \frac{1}{t}\,dt$ for $x \in \mathbb{R}^+$. Hence $\frac{d}{dx}\ln(x) = 1/x$, which shows that ln is differentiable and continuous and also increasing on $\mathbb{R}^+$. Hence ln is one-to-one.
  - We found that $\int 1/x\,dx = \ln|x| + c$ by differentiating $\ln|x|$ as a piecewise function.
  - Algebraic properties of the natural log and how these follow from the definition of ln.
  - The limits of $\ln(x)$ as $x \to \infty$ and $x \to 0^+$. We used these limits and the continuity of ln to argue that $\ln : \mathbb{R}^+ \to \mathbb{R}$ is onto.
  - Logarithmic differentiation. How to use it to find the derivative of a function and in what situations it makes it possible or easier to find the derivative of a function.
  - Since ln is one-to-one and onto it has an inverse function $\mathbb{R} \to \mathbb{R}^+$. We showed that this inverse is e^x and $\frac{d}{dx}e^x = e^x$ and $\int e^x\,dx = e^x + c$.
  - Algebraic properties of e^x and how they follow from the corresponding properties of ln. The limits of e^x as $x \to \infty$ and $x \to -\infty$.

- The general exponential and log functions
  - The general exponential function a^x for $a \in \mathbb{R}^+$. We have shown that $\frac{d}{dx}a^x = a^x \ln(a)$ and $\int a^x\,dx = \frac{a^x}{\ln(a)} + c$. Algebraic properties of a^x and how they follow from the corresponding properties of e^x.
  - The general logarithmic function $\log_a(x)$ for $a \in \mathbb{R}^+ \setminus \{1\}$. Why the general exponential function a^x is invertible (one-to-one and onto). We have shown that $\frac{d}{dx}a^x = \frac{1}{x\ln(a)}$.
  - Algebraic properties of $\log_a$ and how they follow from the corresponding properties of ln.

  - The number e as a limit. We defined e as the value of the inverse function $\exp = \ln^{-1}$ at $x = 1$. We have also shown that $e = \lim_{x\to 0}(1 + x)^{1/x}$ and $e = \lim_{nto\infty}(1 + 1/n)^n$.
- Exponential growth and decay
  - Understand why population growth and radiactive decay are modeled by functions that satisfy the differential equation $\frac{dy}{dt} = k$ and why $y(t) = y(0)e^{kt}$ is a solution to such a differential equation. Solving word problems involving exponential growth/decay.
  - Newton's law of cooling $\frac{dT}{dt} = k(T - T_s)$ and how it can be used to solve problems involving cooling or heating.
  - Compound interest and continuously compounded interest. Continuous interest as a limit compound interest.
- Inverse trigonometric functions
  - Understand how the standard trig functions must be modified so they have inverses and how those inverses can be defined in terms of these modified trig functions.
  - The derivatives of the inverse trig functions.
  - How to use the derivatives of inverse trig functions to integrate certain types of functions.
- Hyperbolic trigonometric functions
  - Definitions of the hyperbolic trig functions in terms of e^x.
  - Hyperbolic trig identities.
  - Derivatives of the hyperbolic trig functions.
  - Defining the inverses of hyperbolic trig functions and expressing them in terms of natural logs.
  - The derivatives of the inverse hyperbolic trig functions, and how those derivatives can be used to integrate certain types of functions.
- L'Hôpital's Rule
  - Statement and conditions of use L'Hôpital's Rule. Intederminate forms.
  - Using l'Hôpital's Rule to evaluate limits of quotients, products, differences, and powers that result in indeterminate forms.
  - Cauchy's Mean Value Theorem and it proof.
  - The proof of l'Hôpital's Rule using Cauchy's Mean Value Theorem in the 0/0 indeterminate case.
- Techniques of integration
  - Integrating by parts. The antiderivatives of ln and the inverse trig functions. Reduction formulas.
  - Trigonometric integrals.
    - Techniques of integrating products and quotients of trig functions by substitution, by parts, or by using reduction formulas.
    - Trigonometric and hyperbolic substitutions.
  - Partial fractions. How to integrate rational functions by expressing them as a sum of partial fractions.
  - Integration tables and computer algebra systems. Using a Table of Integrals to quickly find the indefinite integral of a function. Using substitution to turn a function into an appropriate form to find in the table. Using computer algebra software to evaluate an integral. Elementary functions and integrals that cannot be expressed in terms of elementary functions.
  - Improper integrals. What makes an integral improper. Type 1 and 2 improper integrals. Interpreting and evaluating improper integrals in terms of limits. Convergence and divergence. The Comparison Theorem and how to use it to prove convergence or divergence of an improper integral.

- Applications of integration
  - Finding areas enclosed by curves by integrating along the x-axis or the y-axis.
  - Volumes
    - Finding volumes by dividing into slices.
    - Finding volumes of revolution using cylindrical slices or washers.
    - Finding volumes of revolution using cylindrical shells.
  - Arc length. Definition in terms of integration. Using the arc length formula.
- Sequences and series
  - Definition of a sequence of real numbers. Defining a sequence using a formula or a recursive formula.
  - Formal definition of the limit of a sequence. Convergence and divergence. Definition of $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} a_n = -\infty$.
  - Theorem: If $\{a_n\}$ is a sequence of real numbers and f is a function of real numbers such that $a_n = f(n)$ for all $n \in \mathbb{Z}^+$, and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$. Understand why this is true. Understand why the converse of the theorem is false (counterexample?).
  - The Limit Laws for sequences. Understand why the first three of these are true.
  - The Squeeze Theorem for sequences. Understand why this is true.
  - The Continuity and Convergence Theorem: If $\lim_{n\to\infty} a_n = L$ and f is a function continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$. Understand why this is true and how to use it.
  - Convergence/divergence of the geometric sequence $a_n = r^n$
  - Monotonic sequences. The definition of increasing/decreasing sequence.
  - Bounded sequences. The definition of a sequence bounded above or below.
  - The least upper bound of a set and the greatest lower bound. The Completeness Axiom of real numbers. The Monotonic Sequence Theorem: A sequence that is increasing and bounded above, or decreasing and bounded below is convergent. Can you think of an example of such a sequence?
  - Definition of a series in terms of the partial sums of a sequence. Convergence/divergence and the limit of a series. Examples?
  - The geometric series and under what conditions it is convergent or divergent. The limit of the geometric series.
  - Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Understand why this is true. It is often the contrapositive of this theorem that is used: if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. The book calls this the Test for Divergence. The converse of the theorem is false: the series $\sum_{n=1}^{\infty} a_n$ could be divergent even if $\lim_{n\to\infty} a_n = 0$ is true. The harmonic series is a good example of that.
  - The harmonic series as an example of a divergent series whose terms do approach 0.
  - Limit laws for series: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent then
    - $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
    - $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$
    - $\displaystyle\sum_{n=1}^{\infty}(ca_n) = c\sum_{n=1}^{\infty} a_n$ for all $c \in \mathbb{R}$.
  - The connection between series and improper integrals. The Integral Test. Understand why it holds and how it can be used to prove that a series is convergent or divergent.