MCS 122 Exam 1

1. (10 pts) Use a right-hand sum with a uniform partition to express the area under the curve $y = x^3$ from 0 to 1 as a limit, then evaluate the limit.

Some of the following summation formulas may be useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Let $f(x) = x^3$. If we divide the interval $[0,1]$ into n uniform subintervals, each of those will have width $\Delta x = 1/n$. The subintervals are

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right].$$

Since we are using the right endpoints of these intervals, the sample points are $x_i^* = i/n$. Hence the Riemann sum is

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \frac{1}{n} = \sum_{i=1}^{n} \frac{i^3}{n^4} = \frac{\sum_{i=1}^{n} i^3}{n^4}.$$

We can now use $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ in the numerator:

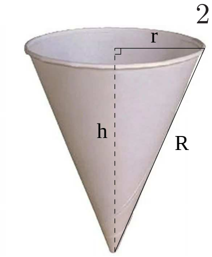
$$R_n = \frac{\frac{n^2(n+1)^2}{4}}{n^4} = \frac{(n+1)^2}{4n^2} = \frac{n^2+2n+1}{4n^2} = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}.$$

Taking the limit as $n \to \infty$,

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) \\
&= \lim_{n\to\infty} \frac{1}{4} + \lim_{n\to\infty} \frac{1}{2n} + \lim_{n\to\infty} \frac{1}{4n^2} \\
&= \frac{1}{4} + \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} + \frac{1}{4} \lim_{n\to\infty} \frac{1}{n^2} \\
&= \frac{1}{4}
\end{aligned}$$

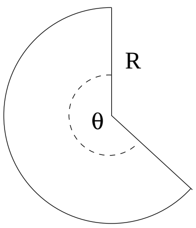
by using $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Hence the area is

$$\int_0^1 x^3\, dx = \frac{1}{4}.$$



2. (10 pts) A (right circular) cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper. Make sure you fully justify your work, including how you know that what you found is an absolute minimum.

   Hint: The volume of a right circular cone of radius r and height h is $V = \frac{\pi r^2 h}{3}$. Since we are constructing a drinking cup, we do not need the base of the cone. Its side can be made of a circular wedge like the one on the left. The radius of the wedge is the same as the slant height of the cone: $R = \sqrt{r^2 + h^2}$, and the central angle is $\theta = \frac{2\pi r}{R}$. The area of such a wedge is given by $A = \frac{\theta R^2}{2}$. You want to find the values of r and h that minimize A.



Unforunately, there was a mistake in the formula $V = \frac{\pi r^3 h}{3}$ I gave you for the volume of a cone. The correct formula is $V = \frac{\pi r^2 h}{3}$. I apologize for this. Fortunately, this makes no difference in how the problem can be solved: the same method of solution works in both cases, only you get somewhat different numbers for the radius and height that minimize the volume. Obviously, I accepted your solution if you used the wrong formula.

Here is the solution using the correct formula first. Since the volume of the cone is supposed to be 27 cm^3, we know

$$27 = \frac{\pi r^2 h}{3} \implies h = \frac{81}{\pi r^2}.$$

We do not need paper for the base of the cone, only for the side, which can be made of the circular wedge on the left, whose area is

$$A = \theta R^2 = \frac{2\pi r}{R} R^2 = 2\pi r R = 2\pi r \sqrt{r^2 + h^2} = 2\pi r \sqrt{r^2 + \frac{6561}{\pi^2 r^4}} = \sqrt{\pi^2 r^4 + \frac{6561}{r^2}}.$$

The area A is now in terms of only the variable r, and so we can use single-variable calculus to find its minimum. Note that since $A \geq 0$, the area A is at its minimum when

$$A^2 = \pi^2 r^4 + \frac{6561}{r^2}$$

is at its mimimum. So let

$$f(r) = \pi^2 r^4 + \frac{6561}{r^2}.$$

We need to find the absolute minimum of $f(r)$ over $r \in (0, \infty)$. We will start by looking for critical points. The derivative

$$f'(r) = 4\pi^2 r^3 - 2\,\frac{6561}{r^3} = 4\pi^2 r^3 - \frac{13122}{r^3}$$

exists for all $r \in (0, \infty)$. So the only kind of critical point is where $f'(r) = 0$.

$$\begin{aligned}
4\pi^2 r^3 - \frac{13122}{r^3} &= 0 \\
4\pi^2 r^3 &= \frac{13122}{r^3} \\
r^6 &= \frac{6561}{2\pi^2} \\
r &= \sqrt[6]{\frac{3^8}{2\pi^2}} = 3\,\frac{\sqrt[3]{3}}{\sqrt[6]{2}\,\sqrt[3]{\pi}} \approx 2.63 \text{ cm.}
\end{aligned}$$

We will show that this critical point is in fact the absolute minimum of $f(r)$. To simplify the notation, let $k = 3\,\frac{\sqrt[3]{3}}{\sqrt[6]{2}\,\sqrt[3]{\pi}}$. First, suppose $r > k$. Then $r^3 > k^3$, and so

$$r^3 > k^3 \implies 4\pi^2 r^3 > 4\pi^2 k^3$$
$$r^3 > k^3 \implies \frac{13122}{r^3} < \frac{13122}{k^3} \implies -\frac{13122}{r^3} > -\frac{13122}{k^3}.$$

Combining these inequalities, we find

$$f'(r) = 4\pi^2 r^3 - \frac{13122}{r^3} > 4\pi^2 k^3 - \frac{13122}{k^3} = 0.$$

So f is an increasing function on (k, ∞), which tells us $f(r) > f(k)$ for $r \in (k, \infty)$.

Similarly, if $0 < r < k$. Then $r^3 < k^3$, and so

$$r^3 < k^3 \implies 4\pi^2 r^3 < 4\pi^2 k^3$$
$$r^3 < k^3 \implies \frac{13122}{r^3} > \frac{13122}{k^3} \implies -\frac{13122}{r^3} < -\frac{13122}{k^3}.$$

Combining these inequalities, we find

$$f'(r) = 4\pi^2 r^3 - \frac{13122}{r^3} < 4\pi^2 k^3 - \frac{13122}{k^3} = 0.$$

So f is a decreasing function on $(0, k)$, which tells us $f(r) < f(k)$ for $r \in (0, k)$. Hence k is an absolute minimum of f by the First Derivative Test for Absolute Extrema. The corresponding height of the cone is

$$h = \frac{81}{\pi k^2} = 3\sqrt[3]{\frac{6}{\pi}} \approx 3.72 \text{ cm}.$$

So to make the cup out of the smallest amount of paper, we would want its radius to be $r = 3\,\frac{\sqrt[3]{3}}{\sqrt[6]{2}\,\sqrt[3]{\pi}} \approx 2.63$ cm and its height to be $h = 3\sqrt[3]{\frac{6}{\pi}} \approx 3.72$ cm.

Here is the same kind of calculation with the wrong formula $V = \frac{\pi r^3 h}{3}$. Since the volume of the cone is supposed to be 27 cm^3, we know

$$27 = \frac{\pi r^3 h}{3} \implies h = \frac{81}{\pi r^3}.$$

We do not need paper for the base of the cone, only for the side, which can be made of the circular wedge on the left, whose area is

$$A = \theta R^2 = \frac{2\pi r}{R}\,R^2 = 2\pi r R = 2\pi r\sqrt{r^2 + h^2} = 2\pi r\sqrt{r^2 + \frac{6561}{\pi^2 r^6}} = \sqrt{\pi^2 r^4 + \frac{6561}{r^4}}.$$

The area A is now in terms of only the variable r, and so we can use single-variable calculus to find its minimum. Note that since $A \geq 0$, the area A is at its minimum when

$$A^2 = \pi^2 r^4 + \frac{6561}{r^4}$$

is at its mimimum. So let

$$f(r) = \pi^2 r^4 + \frac{6561}{r^4}.$$

We need to find the absolute minimum of $f(r)$ over $r \in (0, \infty)$. We will start by looking for critical points. The derivative

$$f'(r) = 4\pi^2 r^3 - 4\,\frac{6561}{r^5} = 4\pi^2 r^3 - \frac{26244}{r^5}$$

exists for all $r \in (0,\infty)$. So the only kind of critical point is where $f'(r) = 0$.

$$\begin{aligned} 4\pi^2 r^3 - \frac{26244}{r^5} &= 0 \\ 4\pi^2 r^3 &= \frac{26244}{r^5} \\ r^8 &= \frac{6561}{\pi^2} \\ r &= \sqrt[8]{\frac{3^8}{\pi^2}} = \frac{3}{\sqrt[4]{\pi}} \approx 2.28 \text{ cm.} \end{aligned}$$

We will show that this critical point is in fact the absolute minimum of $f(r)$. To simplify the notation, let $k = \frac{3}{\sqrt[4]{\pi}}$. First, suppose $r > k$. Combining these inequalities, we find

$$f'(r) = 4\pi^2 r^3 - \frac{26244}{r^5} > 4\pi^2 k^3 - \frac{26244}{k^5} = 0.$$

So f is an increasing function on (k,∞), which tells us $f(r) > f(k)$ for $r \in (k,\infty)$.

Similarly, if $0 < r < k$. Then $r^3 < k^3$ and $r^5 < k^5$, and so

$$\begin{aligned} r < k &\implies r^3 < k^3 \implies 4\pi^2 r^3 < 4\pi^2 k^3 \\ r < k &\implies r^5 < k^5 \implies \frac{26244}{r^5} > \frac{26244}{k^5} \implies -\frac{26244}{r^5} < -\frac{26244}{k^5}. \end{aligned}$$

Combining these inequalities, we find

$$f'(r) = 4\pi^2 r^3 - \frac{26244}{r^5} < 4\pi^2 k^3 - \frac{26244}{k^5} = 0.$$

So f is a decreasing function on $(0,k)$, which tells us $f(r) < f(k)$ for $r \in (0,k)$. Hence k is an absolute minimum of f by the First Derivative Test for Absolute Extrema. The corresponding height of the cone is

$$h = \frac{81}{\pi k^3} = \frac{3}{\sqrt[4]{\pi}} \approx 3.99 \text{ cm.}$$

So to make the cup out of the smallest amount of paper, we would want its radius to be $r = \frac{3}{\sqrt[4]{\pi}} \approx 2.28$ cm and its height to be $h = \frac{3}{\sqrt[4]{\pi}} \approx 2.25$ cm.

3. (5 pts each)
   (a) Find the derivative

$$\frac{d}{dx}\int_{x^2}^{-1} \frac{1}{5+t^3}\,dt.$$

First, let

$$f(x) = \int_{-1}^{x} \frac{1}{5+t^3}\,dt.$$

By the Fundamental Theorem of Calculus,

$$f'(x) = \frac{1}{5+x^3}.$$

Hence

$$\begin{aligned}
\frac{d}{dx}\int_{x^2}^{-1}\frac{1}{5+t^3}\,dt &= \frac{d}{dx}\left(-\int_{-1}^{x^2}\frac{1}{5+t^3}\,dt\right)\\
&= -\frac{d}{dx}f(x^2)\\
&= -f'(x^2)\frac{d}{dx}x^2 && \text{by the chain rule}\\
&= -\frac{2x}{5+x^6}.
\end{aligned}$$

(b) Find the indefinite integral

$$\int \sec^2(x)\sqrt[3]{1+\tan(x)}\,dx.$$

We can evaluate this integral by substituting $u = 1+\tan(x)$. Then $du = \sec^2(x)\,dx$. Hence

$$\int \sec^2(x)\sqrt[3]{1+\tan(x)}\,dx = \int \sqrt[3]{u}\,du = \frac{u^{4/3}}{4/3} + c = \frac{3}{4}\big(1+\tan(x)\big)^{4/3} + c.$$

4. (10 pts) We defined the natural log function $\ln : \mathbb{R}^+ \to \mathbb{R}$ by

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

Use this definition to prove that for all $x, y \in \mathbb{R}^+$,

$$\ln(xy) = \ln(x) + \ln(y).$$

Hint: In class, we started by differentiating both sides.

We will differentiate each side with respect to x, treating y as a constant (or as a variable that is independent of x, if you prefer to think about it that way):

$$\frac{d}{dx}\ln(xy) = \frac{1}{xy}\frac{d}{dx}(xy) = \frac{1}{xy}\,y = \frac{1}{x}$$

and

$$\frac{d}{dx}\big(\ln(x)+\ln(y)\big) = \frac{1}{x} + 0 = \frac{1}{x}.$$

Since $\ln(xy)$ and $\ln(x)+\ln(y)$ have the same derivative on the interval $(0,\infty)$, they can only differ by a constant c according to Corollary 7 in Section 3.2. Hence

$$\ln(xy) = \ln(x) + \ln(y) + c.$$

To find the value of c, we will substitute $x = 1$ and use that

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0.$$

If $x = 1$, then

$$\ln(1\cdot y) = \ln(1) + \ln(y) + c = \ln(y) + c.$$

This shows $c = \ln(y) - \ln(y) = 0$. Therefore

$$\ln(xy) = \ln(x) + \ln(y)$$

for all $x, y \in \mathbb{R}^+$,

5. (5 pts each) **Extra credit problem.**

   (a) Let $f : A \to B$ and $g : B \to C$ be functions where A, B, and C are subsets of the real numbers. Show that if f and g are both one-to-one, then their composition $g \circ f$ is also one-to-one.

   Suppose f and g are one-to-one. Then we know that if $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$, then $x_1 = x_2$, and if $y_1, y_2 \in A$ such that $g(y_1) = g(y_2)$, then $y_1 = y_2$. Let $x_1, x_2 \in A$. Suppose $g \circ f(x_1) = g \circ f(x_2)$. That is $g(f(x_1)) = g(f(x_2))$. Then $f(x_1) = f(x_2)$ because g is one-to-one. It follows that $x_1 = x_2$. We have shown that if $g \circ f(x_1) = g \circ f(x_2)$, then $x_1 = x_2$. Therefore $g \circ f$ is one-to-one.

   (b) Find an example to show that if f and g are one-to-one functions of real numbers, their product fg need not be one-to-one.

   Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x$. It is clear that if $x_1, x_2 \in \mathbb{R}$ are such that $f(x_1) = f(x_2)$, then $x_1 = x_2$. So f is one-to-one. Now, let $g = f$. So g is also one-to-one. Then the function $fg : \mathbb{R} \to \mathbb{R}$ is $fg(x) = f(x)g(x) = x^2$, which is not a one-to-one function. For example, $fg(-2) = (-2)^2 = 2^2 = fg(2)$.