MCS 122 Exam 2 Solutions

1. (10 pts) Use l'Hospital's rule to evaluate the limit
$$\lim_{x\to 0} \frac{\cos(mx)-\cos(nx)}{x^2}.$$
Be sure to explain why using l'Hospital's rule is justified.

By continuity of the cosine at $x=0$,
$$\lim_{x\to 0}\big(\cos(mx)-\cos(nx)\big) = \cos(0)-\cos(0)=0.$$
Also, it is clear that
$$\lim_{x\to 0} x^2 = 0.$$
So the limit in this problem is of the form 0/0. Both the numerator and the denominator are differentiable functions and $\frac{d}{dx}x^2 = 2x \neq 0$ for value of x near 0. By l'Hospital's rule,
$$\lim_{x\to 0}\frac{\cos(mx)-\cos(nx)}{x^2} = \lim_{x\to 0}\frac{-m\sin(mx)+n\sin(nx)}{2x}.$$
This is still of the form 0/0 form since
$$\lim_{x\to 0}\big(n\sin(nx)-m\sin(mx)\big) = n\sin(0)-m\sin(0)=0$$
and
$$\lim_{x\to 0}(2x)=0.$$
by continuity of the sine function and of the polynomial $2x$. Both the numerator and denominator are differentiabkle (we are about to differentiate them) and $\frac{d}{dx}(2x)=2$ is not 0 for x near 0. By l'Hospital's rule,
$$\begin{aligned}\lim_{x\to 0}\frac{n\sin(nx)-m\sin(mx)}{2x} &= \lim_{x\to 0}\frac{n^2\cos(nx)-m^2\cos(mx)}{2}\\ &= \frac{n^2\cos(0)-m^2\cos(0)}{2}\\ &= \frac{n^2-m^2}{2},\end{aligned}$$
where we evaluated the last limit using the continuity of the cosine function. Therefore
$$\lim_{x\to 0}\frac{\cos(mx)-\cos(nx)}{x^2} = \frac{n^2-m^2}{2}.$$

2. (10 pts) In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

Hint: Remember that Newton's Law of Cooling says that if $T(t)$ is the temperature of an object at time t in an environment of constant temperature T_s, then $\frac{dT}{dt} = k(T-T_s)$ for some constant of proportionality k.

Let $y(t) = T(t)-T_s$. Then $y'(t)=T'(t)$. By Newton's Law of Cooling,
$$\frac{dy}{dt} = \frac{dT}{dt} = k(T-T_s) = ky.$$

We know that the solution of the differential equation $\frac{dy}{dt} = ky$ is an exponential function of the form $y(t) = y_0 e^{kt}$. It follows that

$$T(t) = y(t) + T_s = y_0 e^{kt} + T_s$$

where $y_0 = y(0) = T(0) - T_s$ is the initial temperature difference between the body and the environment at time $t = 0$. It is up to us what time we choose to be that initial time $t = 0$. I will choose $t = 0$ to be the time the victim was killed and will measure t in hours. So in our case, $y_0 = 37 - 20 = 17°C$. Therefore

$$T(t) = 17e^{kt} + 20.$$

Let us say the victim died x hours before 1:30 PM. Then $t = x$ at 1:30, and

$$32.5 = T(x) = 17e^{kx} + 20 \implies 17e^{kx} = 12.5.$$

At 2:30

$$30.3 = T(x+1) = 17e^{kx+k} + 20 = 17e^{kx+k} + 20 \implies 17e^{kx+k} = 10.3.$$

Let us divide the second equation by the first:

$$\frac{17e^{kx+k}}{17e^{kx}} = \frac{10.3}{12.5} \implies e^k = \frac{10.3}{12.5}.$$

Hence

$$T(t) = 17e^{kt} + 20 = 17\left(e^k\right)^t + 20 = 17\left(\frac{10.3}{12.5}\right)^t + 20.$$

Substituting this back into the first equation gives

$$12.5 = 17e^{kx} = 17\left(e^k\right)^x = 17\left(\frac{10.3}{12.5}\right)^x \implies \frac{12.5}{17} = \left(\frac{10.3}{12.5}\right)^x.$$

We can find x by taking the natural log of both sides:

$$\ln\left(\frac{12.5}{17}\right) = \ln\left(\frac{10.3}{12.5}\right)^x = x\ln\left(\frac{10.3}{12.5}\right).$$

Hence

$$x = \frac{\ln(12.5/17)}{\ln(10.3/12.5)} \approx 1.6.$$

So the victim died about an hour and 36 minutes before 1:30 PM, that is around 11:54 AM.

3. (10 pts) Let the function $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \sinh(x)$ and note that f is both one-to-one and onto. Define the arcsinh function to be the inverse of f. Prove that

$$\frac{d}{dx}\operatorname{arcsinh}(x) = \frac{1}{\sqrt{1+x^2}}.$$

As usual, we can differentiate both sides of the equation $x = \sinh(\operatorname{arcsinh}(x))$:

$$\frac{d}{dx}x = \frac{d}{dx}\sinh(\operatorname{arcsinh}(x))$$

$$1 = \cosh(\operatorname{arcsinh}(x))\frac{d}{dx}\operatorname{arcsinh}(x)$$

$$\frac{d}{dx}\operatorname{arcsinh}(x) = \frac{1}{\cosh(\operatorname{arcsinh}(x))}$$

To simplify $\cosh(\operatorname{arcsinh}(x)$, we can use the trig identity $\cosh^2(x) - \sinh^2(x) = 1$. First, note that

$$\cosh^2(x) = 1 + \sinh^2(x) \implies \cosh(x) = \pm\sqrt{1 + \sinh^2(x)}.$$

Since $\cosh(x) = \frac{e^x+e^{-x}}{2}$ and e^x and e^{-x} are both positive for any value of x, we know $\cosh(x) > 0$. Hence we can discard the negative square root:

$$\cosh(x) = \sqrt{1 + \sinh^2(x)}.$$

It follows that

$$\cosh(\operatorname{arcsinh}(x) = \sqrt{1 + \sinh^2(\operatorname{arcsinh}(x))} = \sqrt{1 + x^2}.$$

Therefore

$$\frac{d}{dx} \operatorname{arcsinh}(x) = \frac{1}{\cosh(\operatorname{arcsinh}(x))} = \frac{1}{\sqrt{1+x^2}}.$$

4. (10 pts) Use integration by parts to evaluate

$$\int \cos^2(x)\, dx.$$

Hint: It may be wise to check your answer by differentiating it.

Let

$$\int \cos^2(x)\, dx = \int \underbrace{\cos(x)}_{u} \underbrace{\cos(x)\, dx}_{dv}.$$

So $du = -\sin(x)\, dx$ and $v = \sin(x)$. Hence

$$\begin{aligned}
\int \cos(x)\cos(x)\, dx &= \sin(x)\cos(x) - \int \sin(x)\big(-\sin(x)\big)\, dx \\
&= \sin(x)\cos(x) + \int \sin^2(x)\, dx \\
&= \sin(x)\cos(x) + \int 1 - \cos^2(x)\, dx \\
&= \sin(x)\cos(x) + \int 1\, dx - \int \cos^2(x)\, dx \\
&= \sin(x)\cos(x) + x + c - \int \cos^2(x)\, dx
\end{aligned}$$

Hence

$$2 \int \cos(x)\cos(x)\, dx = \sin(x)\cos(x) + x + c.$$

Dividing by 2 gives

$$\int \cos(x)\cos(x)\, dx = \frac{\sin(x)\cos(x) + x + c}{2},$$

or since c is an arbitrary constant anyway

$$\int \cos(x)\cos(x)\, dx = \frac{\sin(x)\cos(x) + x}{2} + c.$$

It is easy to check the answer:

$$\frac{d}{dx}\frac{\sin(x)\cos(x) + x}{2} = \frac{\cos(x)\cos(x) - \sin(x)\sin(x) + 1}{2} = \frac{\cos^2(x) \overbrace{- \sin^2(x) + 1}^{\cos^2(x)}}{2} = \cos^2(x).$$

5. (10 pts) **Extra credit problem.** You have certainly seen functions that are continuous, but not differentiable at a point. A classic example is $f(x) = |x|$ at $x = 0$. It is easy to see that the absolute value function has derivative $f'(x) = 1$ for all $x > 0$ and $f'(x) = -1$ for all $x < 0$. So this function has a legitimate derivative for every value of x near 0, just not at 0.

   Now, suppose f is some function of real numbers and $a \in \mathbb{R}$ such that f is continuous at a, and $f'(x)$ exists for all values of x near a (but not necessarily at a). Show that if $L = \lim_{x\to a} f'(x)$ also exists, then f must be differentiable at a as well, and in fact $f'(a) = L$. So f' is also a continuous function at a.

   Hint: You can do this by using the definition of the derivative for $f'(a)$ and l'Hospital's rule. But make sure you verify that the conditions of l'Hospital's rule are met.

   By definition of the derivative,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

   Since f is continuous at a,

$$\lim_{x\to a} \big(f(x) - f(a)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} f(a) = f(a) - f(a) = 0.$$

   It is also clear that

$$\lim_{x\to a} (x - a) = a - a = 0.$$

   Since f is differentiable at every x near a, so is $f(x) - f(a)$:

$$\frac{d}{dx}\big(f(x) - f(a)\big) = f'(x).$$

   Also

$$\frac{d}{dx}(x - a) = 1,$$

   which is not 0 for any valoue of x near a. Hence we can use l'Hospital's rule to find the limit of the difference quotient:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{f'(x)}{1} = L.$$

   This shows that $f'(a)$ exists, and so f is differentiable at a. In fact,

$$f'(a) = L = \lim_{x\to a} f'(x),$$

   which shows that f' is continuous at a.