MCS 122 Final Exam Solutions

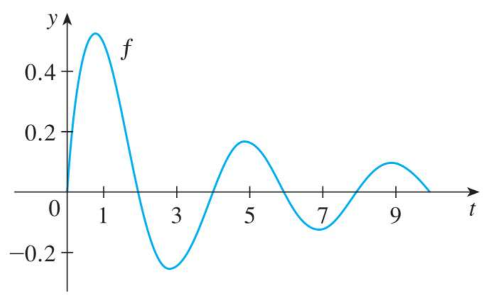
1. (5 pts each)
   (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.

   Let the two roots be $a < b$. Then $f(a) = f(b) = 0$. Since f is differentiable on $\mathbb{R}$, it is also continuous on $\mathbb{R}$. In particular, it is continuous on the interval $[a, b]$ and differentiable on (a, b). So it satisfies the hypotheses of Rolle's Theorem. By Rolle's Theorem, there is a number $c \in (a, b)$ such that $f'(c) = 0$. This shows f' has at least one root at c.

   (b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.

   Let the three roots be $s < t < u$. Since f is differentiable twice on $\mathbb{R}$, both f and its derivative f' are differentiable on $\mathbb{R}$. By the result in part (a), f' must have a root a between s and t and another root b between t and u. Since $a < t < b$, we know f' has two roots $a < b$. By part (a) again, f'' must have a root c between a and b.

2. (5 pts each) Let $g(x) = \int_0^x f(t)\, dt$, where f is the function whose graph is shown.



   (a) At what values of x do the local maximum and minimum values of g occur? Do not forget to justify your answer.

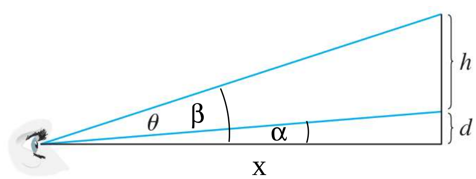
   By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Any local minimum/maximum must be a critical point by Fermat's Theorem. So either $g'(x) = 0$ or $g'(x)$ does not exist there. Based on the graph, $g'(x) = 0$ at $x = 0, 2, 4, 6, 8, 10$. These are the candidates for the local extrema. By the First Derivative Test, the derivative switches from positive to negative at a local maximum and from negative to positive at a local minimum. So $x = 2$ and $x = 6$ are definitely local maxima, and $x = 4$ and $x = 8$ are local minima.
   The graph of f does not cross the x-axis at $x = 0$ and at $x = 10$. Whether these are local extrema depends on what f does outside the interval $[0, 10]$. Since the graph shows no points of f outside this interval, it may well be that the domain of f is $[0, 10]$, and so the domain of g is also $[0, 10]$. If so, then $x = 0$ is actually a local minimum of g, since the value of g will increase as x increases from 0. Hence $g(0) < g(x)$ for every value of x near 0 that is in the domain of g. Similarly, $g(10) > g(x)$ for every value of x near 10 in the domain of g because f is positive and hence g is an increasing function near 10. Hence g has a local maximum at at $x = 10$.
   In conclusion, $g(x)$ has local maxima at $x = 2$ and $x = 6$, and perhaps $x = 10$ depending on what the graph of f does beyond $x = 10$, and local minima at $x = 8$ and $x = 8$, and perhaps $x = 0$ depending on what the graph of f does before $x = 0$.

(b) Where does g attain its absolute maximum value? How can you tell?

The absolute maximum of g is at $x = 2$. It is clear that f is positive on $(0, 2)$, and so g is an increasing function on $[0, 2]$. Hence $g(2) > g(x)$ for all $x < 2$. As x moves past 2, the value of g first decreases, then increases, and so on, as more area between the graph and the x-axis is added to the value of the definite integral $\int_0^x f(t)\,dt$. But notice that the areas under the x-axis are always larger than the areas above the x-axis that follow them. So the value of the integral always decreases by more than it subsequently increases. Hence the value of $g(x) = \int_0^x f(t)\,dt$ always remains less than $g(2)$ for $x > 2$.

3. (10 pts) A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at her eye by the painting?)

Be sure to explain how you know that the value you found for d is indeed the absolute maximum. Remember that a critical point of a function need not be an absolute maximum.



Let x be the distance of the observer from the wall and the angles α and β as indicated in the diagram. Then

$$\alpha = \arctan\left(\frac{d}{x}\right) \qquad \text{and} \qquad \beta = \arctan\left(\frac{d+h}{x}\right).$$

Hence

$$\theta = \beta - \alpha = \arctan\left(\frac{d+h}{x}\right) - \arctan\left(\frac{d}{x}\right).$$

We would like to find the value of $x \in (0, \infty)$ that maximizes θ. So let us differentiate θ with respect to x:

$$\begin{aligned}
\frac{d\theta}{dx} &= \frac{1}{1+\left(\frac{d+h}{x}\right)^2}\frac{d}{dx}\frac{d+h}{x} - \frac{1}{1+\left(\frac{d}{x}\right)^2}\frac{d}{dx}\frac{d}{x} \\
&= \frac{1}{1+\left(\frac{d+h}{x}\right)^2}\left(-\frac{d+h}{x^2}\right) - \frac{1}{1+\left(\frac{d}{x}\right)^2}\left(-\frac{d}{x^2}\right) \\
&= \frac{d}{x^2+d^2} - \frac{d+h}{x^2+(d+h)^2} \\
&= \frac{d(x^2+(d+h)^2) - (d+h)(x^2+d^2)}{(x^2+d^2)(x^2+(d+h)^2)} \\
&= \frac{dx^2 + d(d+h)^2 - (d+h)x^2 - (d+h)d^2)}{(x^2+d^2)(x^2+(d+h)^2)} \\
&= \frac{x^2(d-d-h) + d(d+h)(d+h-d)}{(x^2+d^2)(x^2+(d+h)^2)} \\
&= \frac{h(d^2+dh-x^2)}{(x^2+d^2)(x^2+(d+h)^2)}.
\end{aligned}$$

Note that denominator of this fraction is always positive, and so the derivative exists for all values of x. Hence the only kind of critical point is where the derivative is 0. That will happen when the numerator is 0 and the denominator is not. We know the denominator is never 0. Setting the numerator equal to 0 gives

$$\begin{aligned} 0 &= h(d^2 + dh - x^2) \\ 0 &= d^2 + dh - x^2 && \text{since } h \neq 0 \\ x^2 &= d^2 + dh \\ x &= \pm\sqrt{d(d+h)}. \end{aligned}$$

Since $x \in (0, \infty)$, the only critical point of interest to us is $x = \sqrt{d(d+h)}$. Of course, a critical point need not be a local maximum, let alone an absolute maximum. We would like to show that this critical point is in fact an absolute maximum of $\theta(x)$. Let us look at the sign of the derivative before and after this critical point. First, note that the sign of

$$\frac{d\theta}{dx} = \frac{d^2h + dh^2 - hx^2}{(x^2 + d^2)(x^2 + (d+h)^2)}$$

depends only on the numerator, since the denominator is always positive. Now, there are two possibilities.

If $x > \sqrt{d(d+h)}$, then

$$x^2 > d(d+h) \implies d(d+h) - x^2 < 0 \implies h(d^2 + dh - x^2) < 0$$

since h is positive. It follows that $\frac{d\theta}{dx}$ is negative. So the function $\theta(x)$ is decreasing on the interval $(\sqrt{d(d+h)}, \infty)$.

If $x < \sqrt{d(d+h)}$, then

$$x^2 < d(d+h) \implies d(d+h) - x^2 > 0 \implies h(d^2 + dh - x^2) > 0.$$

It follows that $\frac{d\theta}{dx}$ is positive. So the function $\theta(x)$ is increasing on the interval $(0, \sqrt{d(d+h)})$.

Therefore $\theta(x)$ is always increasing before $x = \sqrt{d(d+h)}$ and is always decreasing afterwards. This shows that θ is at an absolute maximum when $x = \sqrt{d(d+h)}$.

4. (a) (6 pts) Use partial fractions to find

$$\int \frac{2x^2 + 1}{(x^2 + 1)^2}\, dx.$$

Observe that

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{2x^2+2-1}{(x^2+1)^2} = \frac{2(x^2+1)}{(x^2+1)^2} - \frac{1}{(x^2+1)^2} = \frac{2}{x^2+1} - \frac{1}{(x^2+1)^2}.$$

Hence

$$\int \frac{2x^2+1}{(x^2+1)^2}\, dx = \int \frac{2}{x^2+1} - \frac{1}{(x^2+1)^2}\, dx = 2\arctan(x) - \int \frac{1}{(x^2+1)^2}\, dx.$$

To evaluate the second integral, let us substitute $x = \tan(u)$. Then $dx = \sec^2(u)\,du$. Hence

$$\begin{aligned}\int \frac{1}{(x^2+1)^2}\,dx &= \int \frac{1}{(\tan^2(u)+1)^2}\,\sec^2(u)\,du \\ &= \int \frac{1}{(\sec^2(u))^2}\,\sec^2(u)\,du \\ &= \int \cos^2(u)\,du.\end{aligned}$$

By the trig identity $\cos(2u) = \cos^2(u) - \sin^2(u) = 2\cos^2(u) - 1$, we have $\cos^2(u) = \frac{\cos(2u)+1}{2}$, and so

$$\int \cos^2(u)\,du = \int \frac{\cos(2u)+1}{2}\,du = \frac{\sin(2u)}{4} + \frac{u}{2} + c.$$

We can now subsitute x back into this result. Since $x = \tan(u)$, we have $u = \arctan(x)$. Here is one way to express $\sin(2u)$ in terms of x in simplified terms:

$$\begin{aligned}\sin(2u) &= 2\sin(u)\cos(u) \\ &= 2\frac{\sin(u)}{\cos(u)}\,\cos^2(u) \\ &= 2\frac{\tan(u)}{\sec^2(u)} \\ &= 2\frac{\tan(u)}{\tan^2(u)+1} \\ &= 2\frac{x}{x^2+1}.\end{aligned}$$

Hence

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\,\frac{x}{x^2+1} + \frac{\arctan(x)}{2} + c.$$

Putting the pieces together, we get

$$\int \frac{2x^2+1}{(x^2+1)^2}\,dx = \frac{3}{2}\arctan(x) - \frac{x}{2x^2+2} + c.$$

If you do not see right away how to express $\frac{2x^2+1}{(x^2+1)^2}$ as partial fractions, you can use the usual procedure. Set

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Hence

$$\begin{aligned}2x^2+1 &= (Ax+B)(x^2+1) + Cx + D \\ &= Ax^3 + Bx^2 + Ax + B + Cx + D \\ &= Ax^3 + Bx^2 + (A+C)x + (B+D).\end{aligned}$$

Setting corresponding coefficients equal gives

$$\begin{aligned} A &= 0 \\ B &= 2 \\ A + C = 0 &\implies C = -A = 0 \\ B + D = 1 &\implies D = 1 - B = -1. \end{aligned}$$

Hence

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{2}{x^2+1} - \frac{1}{(x^2+1)^2}.$$

(b) (4 pts) Is the improper integral

$$\int_1^\infty \frac{2x^2+1}{(x^2+1)^2}\, dx$$

convergent? If so, find its value.

By our result from part (a),

$$\begin{aligned} \int_1^\infty \frac{2x^2+1}{(x^2+1)^2}\, dx &= \lim_{t\to\infty} \int_1^t \frac{2x^2+1}{(x^2+1)^2}\, dx \\ &= \lim_{t\to\infty} \left[\frac{3}{2}\arctan(x) - \frac{x}{2x^2+2}\right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{3}{2}\arctan(t) - \frac{t}{2t^2+2} - \frac{3}{2}\arctan(1) + \frac{1}{4}\right) \\ &= \frac{3}{2}\lim_{t\to\infty}\arctan(t) - \lim_{t\to\infty}\frac{t}{2t^2+2} - \frac{3}{2}\frac{\pi}{4} \\ &= \frac{3}{2}\frac{\pi}{2} - \lim_{t\to\infty}\frac{\frac{1}{t}}{2+\frac{2}{t^2}} - \frac{3\pi}{8} + \frac{1}{4} \\ &= \frac{3\pi}{4} - \frac{0}{2} - \frac{3\pi}{8} + \frac{1}{4} \\ &= \frac{3\pi}{8} + \frac{1}{4}. \end{aligned}$$

So yes, this improper integral is convergent.

5. (10 pts) Use the method of cylindrical shells to find the volume generated by rotating about the y-axis the region between the graph of $f(x) = \sin(x)$ and the x-axis from $x = 0$ to $x = \pi$.

Using the volume formula with cylindrical shells, we get

$$V = \int_0^\pi 2\pi x \sin(x)\, dx = 2\pi \int_0^\pi x\sin(x)\, dx.$$

We can evaluate this by using integration by parts with $u = x$ and $dv = \sin(x)\, dx$. Then $du = dx$ and $v = -\cos(x)$. So

$$\begin{aligned} \int_0^\pi x\sin(x)\, dx &= -x\cos(x)|_0^\pi - \int_0^\pi -\cos(x)\, dx \\ &= -\pi\cos(\pi) - 0\cos(0) + \sin(x)|_0^\pi \\ &= \pi + \sin(\pi) - \sin(0) \\ &= \pi. \end{aligned}$$

Therefore the volume is $2\pi^2$.

6. (10 pts) Show that the geometric series

$$\sum_{n=0}^{\infty} r^n$$

converges to $1/(1-r)$ if $-1 < r < 1$, and is divergent otherwise.

First, notice that if $|r| \geq 1$, then as $n \to \infty$, r^n either remains 1, or gets arbitrarily large, or jumps back and forth between positive and negative values. In any case, $\lim_{n\to\infty} r^n \neq 0$. Therefore $\sum_{n=0}^{\infty} r^n$ is divergent by Corollary 7 (Test for Divergence) in Section 8.2.

If $-1 < r < 1$, then we can use the algebraic identity

$$(r-1)(r^n + r^{n-1} + \cdots + r + 1) = r^{n+1} - 1,$$

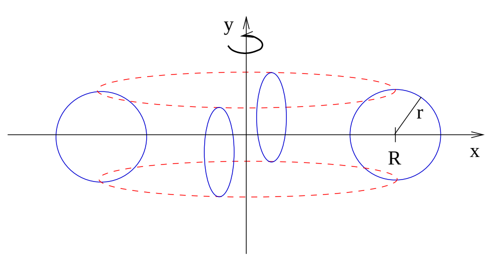
we get

$$r^n + r^{n-1} + \cdots + r + 1 = \frac{r^{n+1} - 1}{r - 1}.$$

Now

$$\begin{aligned}
\sum_{n=0}^{\infty} r^n &= \lim_{n\to\infty} \sum_{0}^{n} r^n \\
&= \lim_{n\to\infty} \frac{r^{n+1} - 1}{r - 1} \\
&= \frac{\lim_{n\to\infty}(r^{n+1} - 1)}{\lim_{n\to\infty}(r - 1)} \\
&= \frac{\lim_{n\to\infty} r^{n+1} - 1}{r - 1} \\
&= \frac{0 - 1}{r - 1} \\
&= \frac{1}{1 - r}.
\end{aligned}$$

7. **Extra credit problem.** The equation of a circle of radius r centered at the point $(R, 0)$ is $(x-R)^2 + y^2 = r^2$. Let $R > r$. When this circle is rotated about the y-axis, the volume it generates is called a *torus*. You know this shape from everyday life as the shape of a donut. The purpose of this exercise is to find the volume of such a torus in two different ways, hopefully obtaining the same result.



(a) (8 pts) Use cylindrical shells with axes parallel to the y-axis to calculate the volume of the torus.

We are going to integrate along the x-axis. It is clear from the diagram that the bounds of the integral are $R - r$ and $R + r$. For $x \in [R - r, R + r]$,

$$(x-R)^2 + y^2 = r^2 \implies y^2 = r^2 - (x-R)^2 \implies y = \pm\sqrt{r^2 - (x-R)^2}.$$

It makes sense that we get two values for y because one is in the lower semicircle and the other is in the upper semicircle. Therefore the height of the cylindrical shell at x is

$$\sqrt{r^2 - (x-R)^2} - (-\sqrt{r^2 - (x-R)^2}) = 2\sqrt{r^2 - (x-R)^2}.$$

Hence the volume of the torus is

$$V = \int_{R-r}^{R+r} 2\pi x \, 2\sqrt{r^2 - (x-R)^2}\, dx = 4\pi \int_{R-r}^{R+r} x\sqrt{r^2 - (x-R)^2}\, dx.$$

We start by substituting $u = x - R$. Then $x = u + R$ and $du = dx$. When $x = R - r$, then $u = -r$, and when $x = R + r$, then $u = r$. Hence

$$\begin{aligned}\int_{R-r}^{R+r} x\sqrt{r^2 - (x-R)^2}\, dx &= \int_{-r}^{r} (u+R)\sqrt{r^2 - u^2}\, du \\ &= \int_{-r}^{r} u\sqrt{r^2 - u^2}\, du + R\int_{-r}^{r} \sqrt{r^2 - u^2}\, du.\end{aligned}$$

Notice that $u\sqrt{r^2 - u^2}$ is an odd function because u is odd and $\sqrt{r^2 - u^2}$ is even. By symmetry,

$$\int_{-r}^{r} u\sqrt{r^2 - u^2}\, du = 0.$$

The second integral can easily be evaluated by noticing that it is the area under a semicircle of radius r, and hence it is $\pi r^2/2$. If you do not notice that, you can evaluate it by using trig substitution. Let $u = r\sin(v)$. Then $du = r\cos(v)\, dv$. When $u = -r$, $v = \arcsin(-1) = -\pi/2$, and when $u = r$, $v = \arcsin(1) = \pi/2$. Therefore

$$\begin{aligned}\int_{-r}^{r} \sqrt{r^2 - u^2}\, du &= \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2\sin^2(v)} r\cos(v)\, dv \\ &= r\int_{-\pi/2}^{\pi/2} \sqrt{r^2(1 - \sin^2(v))}\cos(v)\, dv \\ &= r\int_{-\pi/2}^{\pi/2} \sqrt{r^2\cos^2(v)}\cos(v)\, dv.\end{aligned}$$

Since r is a radius, it must be positive, and so $\sqrt{r^2} = r$. Note that $\cos(v)$ is positive for $v \in [\pi/2, \pi/2]$. So $\sqrt{\cos^2(v)} = \cos(v)$. Now

$$\begin{aligned}r\int_{-\pi/2}^{\pi/2} \sqrt{r^2\cos^2(v)}\cos(v)\, dv &= r^2 \int_{-\pi/2}^{\pi/2} \cos^2(v), dv \\ &= r^2 \left[\frac{\sin(2v)}{4} + \frac{v}{2}\right]_{-\pi/2}^{\pi/2} \\ &= r^2 \left(\frac{\sin(\pi) - \sin(-\pi)}{4} + \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)\right) \\ &= r^2\frac{\pi}{2}.\end{aligned}$$

Putting the pieces together, we can now conlude that the volume of the torus is $4\pi R\pi/2r^2 = 2\pi^2 Rr^2$.

(b) (7 pts) Use the washer method with washers perpendicular to the y-axis to calculate the volume of the torus.

We will now integrate along the y axis. It is clear from the diagram that the bounds of the integral are $-r$ and r. For $y \in [-r, r]$,

$$\begin{aligned}
(x-R)^2 + y^2 &= r^2 \\
(x-R)^2 &= r^2 - y^2 \\
x - R &= \pm\sqrt{r^2 - y^2} \\
x &= R \pm \sqrt{r^2 - y^2}.
\end{aligned}$$

It makes sense that we get two values for x because one is in the inner radius of the washer, and the other is the outer radius. Therefore the volume of the torus using washers is

$$\begin{aligned}
V &= \int_{-r}^{r} \pi\big((R + \sqrt{r^2 - y^2})^2 - (R - \sqrt{r^2 - y^2})^2\big)\, dy \\
&= \pi \int_{-r}^{r} 4R\sqrt{r^2 - y^2}\, dy \\
&= 4R\pi \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy.
\end{aligned}$$

Again, the easiest way to tackle this integral is to recognize that it is the area under a semicircle of radius r, and hence it is $\pi r^2/2$. Otherwise substituting $y = r\sin(u)$ will do the job.

Putting the pieces together, we can now conlude that the volume of the torus is $4R\pi\pi/2r^2 = 2\pi^2 Rr^2$.

The result we got for the volume makes some intuitive sense. Imagine you cut the torus with a plane perpendicular to the major circle and imagine you could now stretch the curved piece straight–it would have to be made of really flexible material. You will get a cylinder whose radius is r and height is $2\pi R$. The volume of such a cylinder is exactly $2\pi r^2 2\pi R = 4\pi^2 Rr^2$.