MCS 150 Exam 1 Solutions

1. Consider the following statement.

   For all real numbers x and y there exists an integer z such that $2z = x + y$.

   (a) (3 pts) Represent the statement as a formula.

$$\forall x, y \in \mathbb{R}, \exists z \in \mathbb{Z}, 2z = x + y$$

   (b) (4 pts) Find the negation, in simplest form, of the formula you answered to part (a).

$$\begin{aligned}\overline{\forall x, y \in \mathbb{R}, \exists z \in \mathbb{Z}, 2z = x + y} &\equiv \exists x, y \in \mathbb{R}, \overline{\exists z \in \mathbb{Z}, 2z = x + y} \\ &\equiv \exists x, y \in \mathbb{R}, \forall z \in \mathbb{Z}, \overline{2z = x + y} \\ &\equiv \exists x, y \in \mathbb{R}, \forall z \in \mathbb{Z}, 2z \neq x + y\end{aligned}$$

   (c) (3 pts) Express the negation in part (b) in words.

   There exist real numbers x and y such that for every integer z, $2z \neq x + y$.

2. (10 pts) Prove that for any odd integer n, the number $2n^2 + 5n + 4$ must be odd.

   Suppose n is an odd integer. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned}2n^2 + 5n + 4 &= 2(2k+1)^2 + 5(2k+1) + 4 \\ &= 8k^2 + 8k + 2 + 10k + 5 + 4 \\ &= 8k^2 + 18k + 11 \\ &= 2(4k^2 + 9k + 5) + 1.\end{aligned}$$

   Since k is an integer, so are k^2, $4k^2$, $9k$, and hence $4k^2 + 9k + 5$. Therefore

$$2n^2 + 5n + 4 = 2(4k^2 + 9k + 5) + 1$$

   is an odd number.

3. (10 pts) Let $x, y \in \mathbb{R}$. Below is Homer Simpson's proof that if $x + y$ is irrational then x and y are irrational. The argument has a mistake in it (D'oh!). Find it and explain why it is a mistake. Is it actually true that if $x + y$ is irrational then x and y are irrational? Do not forget to justify your answer.

   We will prove that if $x + y$ is irrational then x and y are irrational by proving the contrapositive: if x and y are not irrational then $x + y$ is not irrational. So suppose x and y are not irrational. Then they must be rational. Therefore $x = a/b$ and $y = c/d$ for some $a, b, c, d \in \mathbb{Z}$ such that $b, d \neq 0$. Hence

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

   Since $a, b, c, d \in \mathbb{Z}$, the numbers $ad + bc$ and bd must both be integers. We also know that $bd \neq 0$ because b and d are both nonzero. Hence $x + y$ is a rational number. It follows that $x + y$ is not irrational, which is what we wanted to show.

   Homer negated "x and y are irrational" wrong when forming the contrapositive. The correct negation is "x or y is not irrational" or equivalently, "x or y is rational." So the correct contrapositive is: if x or y is not irrational then $x + y$ is not irrational.

It is not actually true that if $x+y$ is irrational then x and y are irrational. For example, if $x=\sqrt{2}$ and $y=0$ then $x+y=\sqrt{2}$, which we know from class is irrational. But it is not true that x and y are both irrational as $y=0$ is clearly rational.

4. (10 pts) Prove by induction that for any $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n}(2i-1)(2i) = 1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \cdots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}.$$

For the base, let $n=1$. Then

$$\sum_{i=1}^{n}(2i-1)(2i) = 1\cdot 2$$

and

$$\frac{n(n+1)(4n-1)}{3} = \frac{1(1+1)(4-1)}{3} = 2$$

are indeed equal.

For the inductive hypothesis, we will assume that

$$\sum_{i=1}^{n}(2i-1)(2i) = \frac{n(n+1)(4n-1)}{3}$$

for some $n \in \mathbb{Z}^+$. We will to prove

$$\sum_{i=1}^{n+1}(2i-1)(2i) = \frac{(n+1)((n+1)+1)(4(n+1)-1)}{3}.$$

By the inductive hypothesis,

$$\begin{aligned}
\sum_{i=1}^{n+1}(2i-1)(2i) &= \underbrace{1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \cdots + (2n-1)(2n)}_{=\frac{n(n+1)(4n-1)}{3}} + (2n+1)(2n+2) \\
&= \frac{n(n+1)(4n-1)}{3} + (2n+1)(2n+2) \\
&= \frac{4n^3+3n^2-n}{3} + \frac{3(4n^2+6n+2)}{3} \\
&= \frac{4n^3+3n^2-n+12n^2+18n+6}{3} \\
&= \frac{4n^3+15n^2+17n+6}{3}
\end{aligned}$$

On right-hand side,

$$\begin{aligned}
\frac{(n+1)((n+1)+1)(4(n+1)-1)}{3} &= \frac{(n+1)(n+2)(4n+3)}{3} \\
&= \frac{(n^2+3n+2)(4n+3)}{3} \\
&= \frac{4n^3+12n^2+8n+3n^2+9n+6}{3} \\
&= \frac{4n^3+15n^2+17n+6}{3}
\end{aligned}$$

Hence

$$\sum_{i=1}^{n+1}(2i-1)(2i) = \frac{(n+1)((n+1)+1)(4(n+1)-1)}{3}.$$

We can now conclude by induction that

$$\sum_{i=1}^{n}(2i-1)(2i) = \frac{n(n+1)(4n-1)}{3}.$$

holds for all $n \in \mathbb{Z}^+$

5. (10 pts) **Extra credit problem.** When we defined even and odd numbers in class, I pointed out that one thing that does not immediately follow from those definitions is that every integer is either even or odd. One way to prove this is by induction. First, prove by induction that every integer $n \geq 0$ is either even or odd. Then extend your result to negative integers by using that every negative integer is -1 times a positive integer.

   For the base case of the induction, let $n = 0$ and note that $0 = 2 \cdot 0$ is an even number. So the statement holds for $n = 0$.

   For the inductive hypothesis, assume that for some $n \in \mathbb{Z}^{\geq 0}$, n is either even or odd. We will show that $n+1$ is either even or odd. First, if n is even then $n = 2k$ for some $k \in \mathbb{Z}$. Hence $n+1 = 2k+1$, which is an odd number. Second, if n is odd then $n = 2k+1$ for some $k \in \mathbb{Z}$. Hence $n+1 = 2k+2 = 2(k+1)$, which is an even number because $k+1$ is an integer.

   By induction, it is true for all $n \in \mathbb{Z}^{\geq 0}$ that n is either even or odd.

   If $n \in \mathbb{Z}^-$ then $-n \in \mathbb{Z}^+$. We have just shown that $-n$ is either even or odd. So either $-n = 2k$ or $-n = 2k+1$ for some $k \in \mathbb{Z}$. In the first case, $n = -2k = 2(-k)$, which is even since $-k \in \mathbb{Z}$. In the second case, $n = -2k-1 = 2(-k-1)+1$, which is odd since $-k-1 \in \mathbb{Z}$.

   This problem was based on a proof Brandon Sisler, one of the students in MCS 150 in fall 2020, came up with when challenged to prove that every integer is even or odd. We had not yet learned about induction at that point, so Brandon basically invented induction for himself to construct his argument.