MCS 150 Exam 2 Solutions

1. (10 pts) The sequence $\{c_n\}_{n=1}^{\infty}$ is defined recursively as

$$\begin{aligned} c_1 &= 3 \\ c_2 &= -9 \\ c_n &= 7c_{n-1} - 10c_{n-2} \qquad \text{for } n \geq 3. \end{aligned}$$

Use (strong) induction to show that $c_n = 4 \cdot 2^n - 5^n$ for all integers $n \geq 1$.

We start with the base cases. If $n = 1$, then

$$4 \cdot 2^n - 5^n = 4 \cdot 2^1 - 5^1 = 3 = c_1.$$

If $n = 2$, then

$$4 \cdot 2^n - 5^n = 4 \cdot 2^2 - 5^2 = -9 = c_2.$$

So the identity indeed holds when $n = 1$ and $n = 2$.

Let us suppose that for some $n \in \mathbb{Z}^{\geq 2}$, $c_k = 4(2^k) - 5^k$ for $k = 1, 2, \ldots, n$. We now want to prove that $c_{n+1} = 4(2)^{n+1} - 5^{n+1}$.

$$\begin{aligned} c_{n+1} &= 7c_n - 10c_{n-1} && \text{by the recurrence} \\ &= 7(4 \cdot 2^n - 5^n) - 10(4 \cdot 2^{n-1} - 5^{n-1}) && \text{by the inductive hypothesis} \\ &= 28 \cdot 2^n - 7 \cdot 5^n - 40 \cdot 2^{n-1} + 10 \cdot 5^{n-1} \\ &= 14 \cdot 2^{n+1} - 7 \cdot 5^n - 10 \cdot 2^{n+1} + 2 \cdot 5^n \\ &= 4 \cdot 2^{n+1} - 5 \cdot 5^n \\ &= 4 \cdot 2^{n+1} - 5^{n+1} \end{aligned}$$

which is what we wanted to prove. It follows by induction that $c_n = 4 \cdot 2^n - 5^n$ for all $n \in \mathbb{Z}^+$.

2. (5 pts each)

(a) Give an example of sets A, B and C such that $A \in B$, and $B \in C$, and $A \notin C$.

Many examples are possible of course. Here is one,

$$\begin{aligned} A &= \{1\} \\ B &= \{A\} = \{\{1\}\} \\ C &= \{B\} = \{\{\{1\}\}\} \end{aligned}$$

It is clear that $A \in B$ and $B \in C$ because that is how we constructed B and C. And $A \notin C$ because the only element in C is $\{\{1\}\}$ which is not equal to A.

(b) Evaluate $\mathcal{P}(\mathcal{P}(\{a, b\}))$.

First,

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Now

$$\begin{aligned} \mathcal{P}(\mathcal{P}(\{a, b\})) &= \mathcal{P}(\{\emptyset, \{a\}, \{b\}, \{a, b\}\}) \\ &= \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \\ &\quad \{\emptyset, \{a, b\}\}, \{\{a\}, \{b\}\}, \{\{a\}, \{a, b\}\}, \{\{b\}, \{a, b\}\}, \\ &\quad \{\{a\}, \{\emptyset, \{a\}, \{b\}\}, \{\emptyset, \{a\}, \{a, b\}\}, \{\emptyset, \{b\}, \{a, b\}\}, \\ &\quad \{\{a\}, \{b\}, \{a, b\}\}, \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\} \end{aligned}$$

3. (10 pts) Define the Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ by $F_0 = 0$, $F_1 = 1$ and the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \in \mathbb{Z}^{\geq 2}.$$

Prove by induction on n that

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$$

for every $n \in \mathbb{Z}^+$.

We start with the base case. If $n = 1$, then

$$F_1F_2 = 1 \cdot 1 = 1 = F_2^2.$$

For the inductive hypothesis, assume

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n-1}F_{2n} = \sum_{i=2}^{2n} F_{i-1}F_i = F_{2n}^2$$

for some $n \in \mathbb{Z}^+$. We want to show

$$\sum_{i=2}^{2(n+1)} F_{i-1}F_i = F_{2n+2}^2$$

By the inductive hypothesis,

$$\begin{aligned}
\sum_{i=2}^{2n+2} F_{i-1}F_i &= \sum_{i=2}^{2n+2} F_{i-1}F_i + F_{2n}F_{2n+1} + F_{2n+1}F_{2n+2} \\
&= F_{2n}^2 + F_{2n}F_{2n+1} + F_{2n+1}F_{2n+2} \\
&= F_{2n}\underbrace{(F_{2n} + F_{2n+1})}_{=F_{2n+2}} + F_{2n+1}F_{2n+2} \\
&= F_{2n}F_{2n+2} + F_{2n+1}F_{2n+2} \\
&= \underbrace{(F_{2n} + F_{2n+1})}_{=F_{2n+2}}F_{2n+2} \\
&= F_{2n+2}^2
\end{aligned}$$

By induction,

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$$

for every $n \in \mathbb{Z}^+$.

4. (10 pts) Let A, B, and C be sets. Prove that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

First, note that $x \in A - (B \cap C)$ iff $x \in A$ and $x \notin B \cap C$. Now, $x \in B \cap C$ iff $x \in B$ and $x \in C$. Hence $x \notin B \cap C$ iff $x \notin B$ or $x \notin C$. So $x \in A$ and $x \notin B \cap C$ iff $x \in A$ and $x \notin B$ or $x \notin C$ iff $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$ (by distributivity of and over or). Note that $x \in A$ and $x \notin B$ iff $x \in A - B$, and similarly, $x \in A$ and $x \notin C$ iff $x \in A - C$. Finally, $x \in A - B$ or $x \in A - C$ iff $x \in (A - B) \cup (A - C)$. Therefore $x \in A - (B \cap C)$ iff $x \in (A - B) \cup (A - C)$, which proves $A - (B \cap C) = (A - B) \cup (A - C)$.

In symbols, we have the following chain of equivalences:

$$\begin{aligned}
x \in A - (B \cap C) &\equiv (x \in A) \wedge (x \notin B \cap C) \\
&\equiv (x \in A) \wedge \overline{x \in B \cap C} \\
&\equiv (x \in A) \wedge \overline{(x \in B) \wedge (x \in C)} \\
&\equiv (x \in A) \wedge (\overline{x \in B} \vee \overline{x \in C}) \\
&\equiv \big((x \in A) \wedge \overline{x \in B}\big) \vee \big((x \in A) \wedge \overline{x \in C}\big) \\
&\equiv \big((x \in A) \wedge (x \notin B)\big) \vee \big((x \in A) \wedge (x \notin C)\big) \\
&\equiv (x \in A - B) \vee (x \in A - C) \\
&\equiv x \in (A - B) \cup (A - C)
\end{aligned}$$

5. (10 pts) **Extra credit problem.** For sets A and B define $A = B$ if A is a subset of B and B is a subset of A. Prove that equality of sets has the following three properties:
   - Reflexivity: for all sets A, $A = A$.
   - Symmetry: for all sets A and B, if $A = B$ then $B = A$.
   - Transitivity: for all sets A, B, and C, if $A = B$ and $B = C$, then $A = C$.

   First, note that any set is a subset of itself. That is $A \subseteq A$ for all sets A. It follows that $A = A$ for all sets A by the definition of equality above.

   Suppose $A = B$. Then $A \subseteq B$ and $B \subseteq A$. So A and B are subsets of each other. Therefore $B = A$.

   Suppose $A = B$ and $B = C$. Then $A \subseteq B$, $B \subseteq A$, $B \subseteq C$, and $C \subseteq B$. By the transitive property of subset inclusion (Theorem 4.2.1), $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$. Similarly, $C \subseteq B$ and $B \subseteq A$ implies $C \subseteq A$. Therefore $A = C$.