MCS 150 Final Exam Solutions

May 11, 2021

1. (5 pts each)
   (a) Represent the statement below as a (logical) formula and express its negation in words. For all real numbers x_1 and x_2, if $x_1^3 + x_1 - 2 = x_2^3 + x_2 - 2$, then $x_1 = x_2$.

   The original statement:
$$\forall x_1, x_2 \in \mathbb{R}, x_1^3 + x_1 - 2 = x_2^3 + x_2 - 2 \implies x_1 = x_2.$$
   The negation: there exist real numbers x_1, x_2 such that $x_1^3 + x_1 - 2 = x_2^3 + x_2 - 2$ and $x_1 \neq x_2$.

   (b) Prove by contrapositive: if $x > 0$ is irrational, then $\sqrt{x}$ is irrational.

   The original statement is that for all $x > 0$, if x is irrational then $\sqrt{x}$ is irrational. The contrapositive is that for all $x > 0$, if $\sqrt{x}$ is not irrational then x is not irrational. So suppose $x > 0$. It is clear that both x and $\sqrt{x}$ are real numbers. Hence if they are not irrational then they must be rational. So we want to prove that if $\sqrt{x}$ is rational then x is also rational. Suppose $\sqrt{x} \in \mathbb{Q}$. Then $x = m/n$ for some $m, n \in \mathbb{Z}$ such that $n \neq 0$. Then
$$x = \sqrt{x}^2 = \frac{m^2}{n^2} \in \mathbb{Q}$$
   since $m^2, n^2 \in \mathbb{Z}$ and $n^2 \neq 0$.

2. (5 pts each)
   (a) Is $\mathbb{Q}^{\geq 0} = \{0\} \cup \mathbb{Q}^+$ a well-ordered subset of $\mathbb{R}$? Why or why not?

   $\mathbb{Q}^{\geq 0}$ is not well-ordered because not every nonempty subset of $\mathbb{Q}^{\geq 0}$ has a smallest element. For example, $\mathbb{Q}^+$ is a nonempty subset of $\mathbb{Q}^{\geq 0}$, but $\mathbb{Q}^+$ does not have a smallest element. This is easy to see by contradiction. Suppose $x \in \mathbb{Q}^+$ is the smallest element. Since x is positive, $x/2$ is also positive. Since x is a rational number, so is $x/2$. Therefore $x/2 \in \mathbb{Q}^+$. It is clear that $x/2 < x$, which contradicts the minimality of x.

   (b) Find the greatest common divisor d of 120 and 615 and express d as an integer linear combination of 120 and 615.

   Using the extended Euclidean algorithm:
$$\begin{aligned} 615 &= 5 \cdot 120 + 15 &&\implies 15 = 1 \cdot 615 - 5 \cdot 120 \\ 120 &= 8 \cdot 15 + 0 &&\implies \gcd(615, 120) = 15 \end{aligned}$$
   So $\gcd(615, 120) = 15 = 1 \cdot 615 - 5 \cdot 120$.

3. (10 pts) Let A and B be sets. Prove that $A \times B = B \times A$ if and only if $A = \emptyset$ or $B = \emptyset$ or $A = B$.

First, we will prove that if $A = \emptyset$ or $B = \emptyset$ or $A = B$, then $A \times B = B \times A$. If $A = \emptyset$, then $A \times B$ and $B \times A$ are both empty, and hence $A \times B = B \times A$. The same is true if $B = \emptyset$. If $A = B$, then $A \times B = A \times A = B \times A$. In all three cases, $A \times B = B \times A$.

Now we will prove that if $A \times B = B \times A$, then $A = \emptyset$ or $B = \emptyset$ or $A = B$ by proving the contrapositive: if it is not true that $A = \emptyset$ or $B = \emptyset$ or $A = B$ then $A \times B \neq B \times A$. Suppose it is not true that $A = \emptyset$ or $B = \emptyset$ or $A = B$. Then $A \neq \emptyset$ and $B \neq \emptyset$ and $A \neq B$. Since $A \neq B$, either A has an element that is not in B or B has an element that is not in A. Without loss of generality, we may assume that it is A that has some element

x that is not in B. Now, since B is not empty, we can choose some element $y \in B$. Then $(x, y) \in A \times B = B \times A$. Therefore $x \in B$ and $y \in A$. This contradicts our assumption that $x \notin B$.

Here is alternate, more direct argument to prove that if $A \times B = B \times A$, then $A = \emptyset$ or $B = \emptyset$ or $A = B$. Suppose $A \times B = B \times A$. If $A = \emptyset$ or $B = \emptyset$, then there is nothing to prove. Suppose A and B are nonempty. Then we can pick a particular element $b \in B$. We will now show that $A \subseteq B$. Let a be any element of A. Then $(a, b) \in A \times B$ and hence $(a, b) \in B \times A$. Therefore $a \in B$ and $b \in A$. It is the first of these that matters to us: $a \in B$. Since this argument applies to every $a \in A$, therefore every $a \in A$ is also an element of B. It follows that $A \subseteq B$. That $B \subseteq A$ can be shown by a symmetric argument, picking a particular element $a \in A$ and letting b run through the elements of B. Hence $A = B$.

4. (a) (4 pts) Define the intersection and the union of two sets A and B.

   The intersection of A and B is the set
   $$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$
   The union of A and B is the set
   $$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

   (b) (6 pts) Prove that the intersection is distributive over the union:
   $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
   for all sets A, B, and C.

   First note that
   $$x \in A \cap (B \cup C) \iff (x \in A) \wedge (x \in B \cup C) \iff (x \in A) \wedge \big((x \in B) \vee (x \in C)\big).$$
   Now, if $x \in A$, and $x \in B$ or $x \in C$ then either $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. So either $x \in A \cap B$ or $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$. So if $x \in A \cap (B \cup C)$ then $(A \cap B) \cup (A \cap C)$. It follows that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
   This works the other way too. If $x \in (A \cap B) \cup (A \cap C)$ then either $x \in A \cap B$ or $x \in A \cap C$. Hence either $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. In both cases, $x \in A$. So $x \in A$, and $x \in B$ or $x \in C$, which is equivalent to $x \in A \cap (B \cup C)$. So if $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap (B \cup C)$. It follows that $A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
   We can now conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

5. (10 pts) Use strong induction to prove that every integer $n \geq 2$ can be expressed as a product of prime numbers.

The base case of the induction is $n = 2$. Since 2 is a prime, it is already expressed as a product of primes (with a single factor). For the inductive hypotheses, assume that for some $n \in \mathbb{Z}^{\geq 2}$, every integer $k = 2, 3, \ldots, n$ can be expressed as a product of primes. We will show that $n + 1$ can be expressed as a product of primes.

If $n+1$ is prime then it is already expressed as a product of primes. Otherwise $n+1$ must be composite, and hence $n + 1 = ab$ for some $a, b \in \mathbb{Z}^+$ such that $a, b \neq 1, n + 1$. Clearly, $a, b > 1$, and so
$$1 < a \implies b < ab = n + 1.$$
Similarly,
$$1 < b \implies a < ab = n + 1.$$

Hence $a, b \leq n$. By the inductive hypothesis, both a and b can be expressed as products of primes. Multiplying those two products together gives a product of primes that is equal to $ab = n + 1$.

6. (5 pts each) Here is a problem about fairy tail combinatorial challenges.

(a) When the seven dwarfs descend to their gemstone mine, they ride in a train that has two mine carts. Each cart can fit up to four dwarfs. How many ways are there for the seven dwarfs to get into the two carts? Keep in mind that the mine carts are not identical as one is in the front of the train and the other is in the back.

A good way to look at this is to pretend that there are eight dwarfs, let us say that the eighth one is called Nil. Once the eight dwarfs have gotten into the mine carts, Nil promptly vanishes in a puff of smoke. This way, one of the carts ends up having four dwarfs on it, while the other ends up having three. So, if we have eight dwarfs, there are $\binom{8}{4}$ different ways of choosing the four that sit in the front cart. The rest of them sit in the cart in the back with no additional choices remaining. So there are

$$\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$$

ways for the seven dwarfs to get into the two carts.

Here is an alternative way to count the different seating arrangement. There are either three dwarfs in the from cart and four in the back, or four in the front and three in the back. The set of arrangements with three dwarfs in the front and four in the back and the set of arrangements with four dwarfs in the front and three in the back are mutually disjoint. If there are three dwarfs in the front, there are $\binom{7}{3}$ to pick them from among the seven dwarfs. Then the other four dwarfs get in the back cart without any additional choices. If there are three dwarfs in the back, there are $\binom{7}{3}$ to pick the ones that sit in the back from among the seven dwarfs. Then the other four dwarfs get in the front cart without any additional choices. So there are $\binom{7}{3}$ of each kind of arrangement. As we already noted, these are mutually exclusive, so the total number of different seating arrangements is

$$2\binom{7}{3} = 2\,\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 70.$$

Finally, note that we got the same number thinking about the problem in two different ways, which is always reassuring.

(b) Snow White and the seven dwarfs are sitting down to dinner. There are two round tables and four chairs at each table. Both the tables and the chairs are identical to one another.

"How many different ways can the eight of us sit down around the two tables?" challenges Snow White her little friends to some combinatorial thinking.

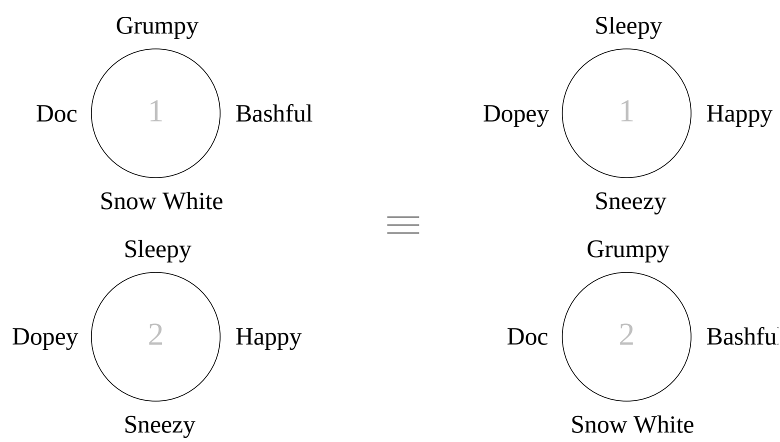
"Well, there are eight ways to pick the first person that sits down," replies Happy. "Then there are seven ways to pick the one sitting to the first person's right. Then six ways to pick the person sitting to the second person's right. Then five ways to pick the person to the third person's right. That is $8 \cdot 7 \cdot 6 \cdot 5$ ways so far."

"Wait," says Grumpy. "Those four already sitting could now all stand up and move one seat to the right without really changing their seating arrangement. Or they could move two seats to the right, or three seats to the right. There are really four equivalent seating arrangement because of such rotations. So we need to divide this by 4."

"Yes, of course!" acknowledges Sneezy.

"Now, we have four ways to pick the first person that sits at the other table," continues Sleepy. "Then three ways to pick the person sitting on their right, two ways for the person sitting on their right, and finally, there is only person left to occupy the last remaining seat. But I know, we need to divide this by 4 as well because of the four equivalent seating arrangements that result from rotations around the table."
"So the number of different seating arrangements is $\frac{8\cdot7\cdot6\cdot5}{4}\frac{4\cdot3\cdot2\cdot1}{4}$," concludes Doc.
"Which is $8!/4^2$, or just $7!/2$," adds Bashful.
"You are close, but not quite right," says Snow White.
What mistake did Snow White object to in the dwarfs' thinking? What is the correct number of different seating arrangements?

The dwarfs' mistake is that they do not take into account that the two tables are identical. So if those sitting at one of the tables traded seats with those sitting at the other table (without altering the order in which they sit at each table), that would result in an equivalent arrangement. For example, the two seating arrangements below are equivalent.



Since the tables are not actually numbered, there is really no difference between the two arrangements in the diagram. Therefore the dwarfs counted the same arrangement twice. Everything else they said was right. So the correct number of different seating arrangements is

$$\frac{1}{2}\frac{7!}{2} = \frac{7!}{4} = 1260.$$

7. **Extra credit problem.** In class, we gave two different arguments that $|\mathcal{P}(A)| = 2^{|A|}$ for any finite set A. It is also possible to prove this by induction on the cardinality $n = |A|$. The base case is $n = 0$. If $n = 0$ then $A = \emptyset$, so $\mathcal{P}(A) = \{\emptyset\}$ and hence $|\mathcal{P}(A)| = 1 = 2^0 = 2^{|A|}$. For the inductive hypothesis, assume that for some $n \in \mathbb{Z}^{\geq 0}$ if A is a set of cardinality n then $|\mathcal{P}(A)| = 2^n$. Your job is to do the inductive step: prove that if A is any set of cardinality $n+1$ then $|\mathcal{P}(A)| = 2^{n+1}$. I will break this down into a few steps for you.
   (a) (2 pts) Suppose $|A| = n + 1$. Choose a particular element $x \in A$. Let $B = A - \{x\}$. How many elements does B have? How many subsets does B have? Why?

   Since x was an element in A, the set B has one less element than A. So $|B| = n$. Hence B has 2^n subsets by the inductive hypothesis.

   (b) (6 pts) Note that if $S \subseteq B$ then S is also a subset of A, and $S \cup \{x\}$ is also a subset of A. Let

$$C = \{S \mid S \subseteq B\} \cup \{S \cup \{x\} \mid S \subseteq B\}.$$

Then C is a collection of subsets of A. How many elements does C have? Do they all have to be different? Why?

It is clear that that all 2^n subsets of B are elements of S. Then for every subset S of B, $S \cup \{x\}$ is also an element of C. Since there are 2^n subsets of B, there should be 2^n subsets of the form $S \cup \{x\}$. But are they all different from each other? Suppose S and S' are two different subsets of B. Could $S \cup \{x\}$ and $S' \cup \{x\}$ be the same? If S and S' are different, then one of them must have an element that is not in the other. Without loss of generality, let us say $y \in S$ but $y \notin S'$. Then $y \in S \cup \{x\}$ too. Note that $y \neq x$ because $y \in S \subseteq B$, so $y \in B$ but $x \notin B$. So y is not an element of $S' \cup \{x\}$ either. Therefore if S and S' are different, then $S \cup \{x\}$ and $S' \cup \{x\}$ must also be different. Finally, observe that if $S, S' \subseteq B$ then $S \neq S' \cup \{x\}$ because $x \notin S$. So C has 2^n elements of the form $S \subseteq B$ and another 2^n elements of the form $S \cup \{x\}$ where $S \subseteq B$. Hence C has $2(2^n) = 2^{n+1}$ distinct elements.

(c) (5 pts) We already know that every element of C is a subset of A. Show that every subset of A is an element in C, that is if $T \subseteq A$ then $T \in C$.

Let $T \subseteq A$. If $x \notin T$ then T is also a subset of B and hence $T \in C$. If $x \in T$, then let $S = T - \{x\}$. Clearly, $T = S \cup \{x\}$. Since $S \subseteq T$, $S \subseteq A$. Notice that it is also true that $S \subseteq B$ because $x \notin B$ and every other element of A is also in B. So T is of the form $S \cup \{x\}$ for a subset S of B. Hence T is an element of C in this case too.

(d) (2 pts) Conclude that $\mathcal{P}(A) = C$ and $|\mathcal{P}(A)| = 2^{n+1}$.

We know that every element of C is a subset of A and every subset of A is an element of C. So $C = \mathcal{P}(A)$. We also know that $|C| = 2^{n+1}$. Hence $|\mathcal{P}(A)| = 2^{n+1}$.