MCS 150 Exam 1 Solutions

Oct 6, 2017

1. (5 pts each) Let A and B be sets. Of the following statements, which are true and which are false? If you think a statement is true, find a convincing argument to show it is true; if not, find an argument or a counterexample to show it is false.

   (a) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

   This is true. Note that
   $$(A \cup B) \setminus (A \cap B) = \{x \mid x \in A \text{ or } x \in B \text{ and } x \notin A \cap B.$$
   Now $x \in A \cap B$ means $x \in A$ and $x \in B$. So $x \notin A \cap B$ if it is not true that $x \in A$ and $x \in B$, that is either $x \notin A$ or $x \notin B$. So $x \in (A \cup B) \setminus (A \cap B)$ if x is in A or B but not in the other. So either $x \in A$ but $x \notin B$, or $x \in B$ but $x \notin A$. Another way to write this is $x \in A \setminus B$ or $x \in B \setminus A$. That is $x \in (A \setminus B) \cup (B \setminus A)$. Hence
   $$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

   (b) $(A \setminus B) \setminus A = A \setminus (B \setminus A)$

   This is false. For example, let $A = \{1\}$ and $B = \{2\}$. Then $A \setminus B = \{1\}$ and $(A \setminus B) \setminus A = \emptyset$. Whereas $B \setminus A = \{2\}$ and $A \setminus (B \setminus A) = \{1\}$. These are not equal, so in general,
   $$(A \setminus B) \setminus A \neq A \setminus (B \setminus A).$$

2. (5 pts each)

   (a) Let x, y be odd integers. Prove that $4 \nmid (x^2 + y^2)$.

   Since x and y are odd, $x = 2k + 1$ and $y = 2n + 1$ for some $k, n \in \mathbb{Z}$. Therefore
   $$\begin{aligned} x^2 + y^2 &= (2k+1)^2 + (2n+1)^2 \\ &= 4k^2 + 4k + 1 + 4n^2 + 4n + 1 \\ &= 4(k^2 + k + n^2 + n) + 2. \end{aligned}$$
   Since $k^2 + k + n^2 + n$ is an integer, this number has a remainder of 2 when divided by 4. Hence $x^2 + y^2$ is not divisible by 4.

   (b) Let $a, b, c \in \mathbb{Z}$. Show that if $a|b$ and $a|c$ then $a|kb + nc$ for any $k, n \in \mathbb{Z}$.

   If $a|b$ and $b|c$ then $b = xa$ and $c = ya$ for some $x, y \in \mathbb{Z}$. So
   $$kb + nc = k(xa) + n(ya) = (kx)a + (ny)a = (kx + ny)a.$$
   Note that $kx + ny$ must be integer, so $a|c$.

3. (5 pts each)

   (a) Use the Euclidean Algorithm to find the greatest common divisor of 1551 and 2021.

   $$\begin{aligned} 2021 &= 1551 + 470 \\ 1551 &= 3 \cdot 470 + 141 \\ 470 &= 3 \cdot 141 + 47 \\ 141 &= 3 \cdot 47 + 0 \end{aligned}$$

   So according to the Euclidean Algorithm, $\gcd(1551, 2021) = 47$.

(b) Write the greatest common divisor you found in part (a) as an integer linear combination of 1551 and 2021.

Backtracking through the algorithm in part (a), we can write

$$\begin{aligned}
47 &= 470 - 3 \cdot 141 \\
&= 470 - 3 \cdot (1551 - 3 \cdot 470) \\
&= 10 \cdot 470 - 3 \cdot 1551 \\
&= 10 \cdot (2021 - 1551) - 3 \cdot 1551 \\
&= 10 \cdot 2021 - 13 \cdot 1551
\end{aligned}$$

4. (10 pts) Let $a, b, c \in \mathbb{Z}$ such that $a|(bc)$. Prove that if a and b are relatively prime then $a|c$.

Let $a, b, c \in \mathbb{Z}$ such that $a|(bc)$. Then $bc = ka$ for some $k \in \mathbb{Z}$. If a and b are relatively prime, then $\gcd(a, b) = 1$. By Bezout's Identity (or the extended Euclidean Algorithm), we can write $1 = ma + nb$ for some $m, n \in \mathbb{Z}$. Multiply both sides by c to get

$$c = (ma + nb)c = mac + nbc = mac + n(ka) = (mc + nk)a.$$

Hence $a|c$.

5. (10 pts) **Extra credit problem.** Let $a, b \in \mathbb{Z}$ both not 0. Let m be a least common multiple of a and b. Then m/a and m/b must be integers. Prove that m/a and m/b are relatively prime.

We will use three results to do this. We proved in class that if $d = \gcd(a, b)$ then $d = xa+yb$ for some $x, y \in \mathbb{Z}$. We also showed in class that $m = (ab)/d$. Hence $m/a = b/d$ and $m/b = a/d$. Finally, one of your homework problems was to prove that if you can write 1 as an integer linear combination of two integers then those two integers are relatively prime. Notice that $m/a = b/d$ and $m/b = a/d$. Now dividing $d = xa + yb$ by d gives

$$1 = \frac{xa + yb}{d} = x\frac{a}{d} + y\frac{b}{d} = x\frac{m}{b} + y\frac{m}{a}.$$

Since 1 is an integer linear combination of m/a and m/b, these two numbers must be relatively prime.

Here is alternate argument. Let $d = \gcd(m/a, m/b)$. Then $m/a = xd$ and $m/b = yd$ for some $x, y \in \mathbb{Z}$. So $m/d = xa$ and $m/d = yb$. This shows that m/d is a common multiple of a and b. Since m is a least common multiple of a and b it must divide any other common divisor. Hence $m|(m/d)$. So there exists $k \in \mathbb{Z}$ such that $m/d = km$. Now, $m = (dk)m$ and since k and d are both integer and $d > 0$, the only way this can happen is that $d = k = 1$.