MCS 150 Homework 4

1. Let x, y be odd integers. Use modular arithmetic modulo some appropriate $n \in \mathbb{Z}$ to do the following.
   (a) Prove that $4 | (x^2 - y^2)$.
   (b) Prove that $8 | (x^2 - y^2)$.
   (c) Prove that $4 \nmid (x^2 + y^2)$.
2. Middle school students are often taught that they can check whether a positive integer is divisible by 9 by adding its digits together and checking if the sum is divisible by 9. E.g. to check if $9|84672$, we can check if $8+4+6+7+2 = 27$ is divisible by 9, which it is. Prove that if n is any positive integer, then dividing it by 9 gives the same remainder as dividing the sum of its digits by 9. (Hint: write n as $d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_1 10 + d_0$ where $d_k, \ldots, d_0$ are the digits of n and work modulo 9.)
3. (a) Let $p = 7$ and $x = 3$. Write down the numbers $x, 2x, \ldots, 6x$ and reduce each down modulo 7 to a remainder between 0 and 6. What do you notice?
   (b) Does this work for any other integer x that is not divisible by 7?
   (c) Let p be a prime number and $x \in \mathbb{Z}$ such that $p \nmid x$. Show that $x, 2x, \ldots, (p-1)x$ are all distinct modulo p, i.e. no two of them are congruent modulo p. Conclude that
   $$\{x, 2x, \ldots, (p-1)x\} \equiv \{1, 2, \ldots, p-1\} \pmod{p},$$
   that is to say these two sets contain the same elements modulo p.