MCS 150 Homework 7

1. Let $n \in \mathbb{Z}^+$. In Homework 5, you noticed that if $n = p^k$ for some prime number p and $k \in \mathbb{Z}^+$, then the number of units among the canonical representatives $0, 1, \ldots, n-1$ is $p^k - p^{k-1}$.
   (a) Now suppose n is any integer, i.e. not necessarily prime or a prime power. Play with some examples until you can formulate a conjecture about the number of units among the canonical representatives $0, 1, \ldots, n-1$ modulo n.
   (b) Let $m, n \in \mathbb{Z}^+$ such that m and n are relatively prime. Let M be the number of units among the canonical representatives $0, 1, \ldots, m-1$ modulo m and let N be the number of units among the canonical representatives $0, 1, \ldots, n-1$ modulo n. Prove that the number of units among the canonical representatives $0, 1, \ldots, mn-1$ modulo mn is MN.
   (c) Prove your conjecture from part (a). (Hint: the result you proved in part (b) should be very helpful.)
2. Another divisibility test you may be familiar with is that you can check whether a positive integer is divisible by 11 by alternately adding and subtracting its digits and checking if the result is divisible by 11. E.g. to check if $11|73854$, we can check if $4-5+8-3+7 = 11$ is divisible by 11, which it is. Prove that if n is any positive integer, then dividing it by 11 gives the same remainder as dividing such an alternating sum of its digits (starting with the rightmost digit) by 11. (Hint: write n as $d_k 10^k + d_{k-1}10^{k-1} + \cdots + d_1 10 + d_0$ where $d_k, \ldots, d_0$ are the digits of n and work modulo 11.)
3. Let $n \in \mathbb{Z}^+$. In class, we defined the Euler ϕ function by letting $\phi(n)$ be the number of integers k between 1 and n that are relatively prime to n. Remember that this is also the number of units modulo n among the canonical representatives $0, 1, \ldots, n-1$. Another name ϕ is known by is the *totient function*. We showed that if $n = p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l}$ where $p_1, \ldots, p_l$ are the prime factors of n, then
$$\phi(n) = \left(p_1^{k_1} - p_1^{k_1-1}\right)\left(p_2^{k_2} - p_2^{k_2-1}\right) \cdots \left(p_l^{k_l} - p_l^{k_l-1}\right).$$
   (a) Choose some examples for n and find $\phi(n)$.
   (b) Is it true that if $m \leq n$ then $\phi(m) \leq \phi(n)$?
   (c) Prove that
$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_l}\right).$$