MCS 150 Homework 9

1. An unusual operation on nonnegative binary (base 2) integers is the exclusive or operation. In computer science, it is often written as XOR, in math the $\oplus$ sign is sometimes used. It is defined by the following rule: $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$. On multidigit numbers, this is done by doing it on corresponding digits. E.g. $110101_2 \oplus 10010_2 = 100111_2$. Notice that this is as if someone tried to add the two numbers but forgot to carry. We will now explore some of the properties of $\oplus$.
   (a) Is $\oplus$ commutative? That is, is it true that $x \oplus y = y \oplus x$ for any $x, y \in \mathbb{Z}^{\geq 0}$?
   (b) Is $\oplus$ associative? That is, is it true that $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for any $x, y, z \in \mathbb{Z}^{\geq 0}$?
   (c) What can you say about $x \oplus x$?
   (d) Is there an identity with respect to $\oplus$? That is, is there a number $i \in \mathbb{Z}^{\geq 0}$ such that $x \oplus i = x$ and $i \oplus x = x$ for every $x \in \mathbb{Z}^{\geq 0}$?

2. Let $x, y \in \mathbb{R}$.
   (a) Prove that if x and y are both rational numbers then $x + y$, $x - y$, xy are all rational numbers, and that if $y \neq 0$ then x/y is also rational.
   (b) If x is a rational number and y is an irrational number is it true that $x + y$ and xy must be irrational?
   (c) If x and y are both irrational numbers is it true that xy must be irrational?
   (d) If xy is an irrational number is it true that at least one of x and y must be irrational?

3. Prior to the age of Pythagoras (around the 5th century B.C.), ancient Greek mathematicians had assumed that every number could be written as a ratio of two integers, i.e. every number was rational. Tradition attributes the discovery that $\sqrt{2}$ cannot be written as a ratio of two integers to a member of Pythagoras's school by the name of Hippasus. This was apparently great shock to his fellow Pythagoreans. They may have been able to deal with it if Hippasus had kept quiet about his discovery. But he did not. The other Pythagoreans were so upset by his indiscretion of letting such an inconvenient truth become public knowledge that they put him on a boat, took him out to sea and threw him in the water to drown. None of this is historical fact, more just part of mathematical folklore, but it is interesting anyway. Whether any of this is true, it is known that the Pythagoreans' original proof of the irrationality of $\sqrt{2}$ was based on a geometric argument comparing the side and the diagonal of a square. You are probably familiar with the following algebraic argument.

   Suppose that $\sqrt{2} = m/n$ where $m, n \in \mathbb{Z}$. Without loss of generality, we may assume that m and n are relatively prime. If they are not, just divide them by their greatest common divisor. Now
   $$\sqrt{2}^2 = \frac{m^2}{n^2} \implies 2n^2 = m^2.$$
   Since $2n^2$ is obviously divisible by 2, m^2 must also be divisible by 2. But 2 is prime, so by Euclid's Lemma, $2|m$. Hence $m = 2k$ for some $k \in \mathbb{Z}$. Now
   $$2n^2 = (2k)^2 \implies 2n^2 = 4k^2 \implies n^2 = 2k^2.$$
   Since $2k^2$ is obviously divisible by 2, n^2 must also be divisible by 2. But 2 is prime, so by Euclid's Lemma, $2|n$. But this contradicts that m and n are supposed to be relatively prime.

   Use a similar argument to show that if p is any prime number, $\sqrt{p}$ is irrational. Can you think of a way to generalize this to composite numbers $x \in \mathbb{Z}^{+}$?