MCS 150 Exam 1 Solutions

1. (5 pts each) Consider these propositional functions:

$$p(n) : n \text{ is prime}$$
$$q(n) : n \text{ is even}$$
$$r(n) : n > 2$$

Express these formulas in words:

(a) $\exists n \in \mathbb{Z}\ \big[p(n) \wedge \big(q(n) \vee r(n)\big)\big]$
(Hint: Take care to make sure your sentence is not ambiguous.)

There is an integer that is prime and either even or greater than 2.

Note that saying "there is an integer that is prime and even or greater than 2" is ambiguous because it could also mean

$$\exists n \in \mathbb{Z}\ \big[\big(p(n) \wedge q(n)\big) \vee r(n)\big].$$

Using either-or clears up this ambiguity. It is not the only way to do it, you could also use a comma before the "and," but either-or is a slick tool in English to deal with such issues.
Also note that writing "there is an integer n such that n is prime and either n is even or n is greater than 2" is entirely correct, but is a more complicated sentence, and including the references to n does not add anything to clarify the meaning. Precision is important in math, but conciseness is also valued.

(b) $\forall n \in \mathbb{Z}\ \big[\big(p(n) \wedge r(n)\big) \implies \overline{q(n)}\big]$

Every prime greater than 2 is odd.
Again, saying "for all integers n if n is prime and n is also greater than 2 then n is not even" is correct, but doesn't the first sentence sound so much more natural?

2. (10 pts)

(a) Let x be a positive real number. Prove by contrapositive: if x is an irrational number then $\sqrt{x}$ is irrational.

We will prove the contrapositive: if $\sqrt{x}$ is rational then x is also rational. So suppose $\sqrt{x} \in \mathbb{Q}$. Then $x = m/n$ for some integers m and $n \neq 0$. Since x is positive, $x = \sqrt{x}^2$. Hence

$$x = \sqrt{x}^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}.$$

As m and n are integers, so are m^2 and n^2. We also know $n \neq 0$, and therefore $n^2 \neq 0$. It follows that x is a rational number.

An even quicker way to do this would be to use the result from class that the product of two rational numbers is rational. Then it immediately follows that if $\sqrt{x} \in \mathbb{Q}$, then $x = \sqrt{x}\sqrt{x}$ is also rational.

(b) Apply the result in part (a) to show that $\sqrt[4]{2}$ is irrational, using the assumption that $\sqrt{2}$ is irrational.

We proved in class that $\sqrt{2}$ is irrational, so we do not even need to assume this. By the proposition in part (a), $\sqrt[4]{2} = \sqrt{\sqrt{2}}$ must also be irrational.

3. (a) (4 pts) Let $x, y \in \mathbb{R}$. Is it true that $(x+y)^2 = x^2 + y^2$ if and only if $x = 0$ and $y = 0$?

It is true that if $x = 0$ and $y = 0$ then

$$(x+y)^2 = (0+0)^2 = 0 = 0^2 + 0^2 = x^2 + y^2.$$

But the converse, that if $(x+y)^2 = x^2 + y^2$ then $x = 0$ and $y = 0$, is false. For example, if $x = 2$ and $y = 0$ then $(x+y)^2$ and x^2+y^2 are both equal to 4. Hence the biconditional above is false.

(b) (6 pts) Find the converse, inverse, and contrapositive of the implication below.

If the integer number n is a perfect square then n is either odd or it is a multiple of 4.

Converse: if the integer number n is either odd or it is a multiple of 4 then n is a perfect square.
Inverse: if the integer number n is not a perfect square then n is neither odd nor a multiple of 4.
Contrapositive: if the integer number n is neither odd nor a multiple of 4 then n is not a perfect square.

Note that the part about "integer number" is not really a part of the conditional statement. It is an implicit universal quantifier, which says that the conditional statement applies to every integer number n. Therefore the right place to put it in the converse and the contrapositive is also the beginning of the sentence. I would not say that "if the n is either odd or it is a multiple of 4 then the integer number n is a perfect square" is incorrect, but it definitely seems awkward to only reveal near the end of the sentence that n is supposed to be an integer.

4. (10 pts) Let $n \in \mathbb{Z}$. Prove that n^2 is divisible by 4 if and only if n is even.

Suppose n is even. Then $n = 2q$ for some $q \in \mathbb{Z}$. So

$$n^2 = (2q)^2 = 4q^2.$$

Since q is an integer, so is q^2 and therefore n^2 is a multiple of 4.

We will prove the contrapositive of the other direction: if n is not even then n^2 is not divisible by 4. So suppose n is not even. Then it is odd and $n = 2q + 1$ for some $q \in \mathbb{Z}$. Hence

$$n^2 = (2q+1)^2 = 4q^2 + 4q + 1.$$

It seems fairly obvious that $4q^2 + 4q + 1$ cannot be a multiple of 4, but we need to prove that. Suppose that it were a multiple of 4. Then $4q^2 + 4q + 1 = 4r$ for some $r \in \mathbb{Z}$ and hence

$$r = \frac{4q^2 + 4q + 1}{4} = q^2 + q + \frac{1}{4},$$

which implies

$$\frac{1}{4} = r - q^2 - q.$$

Since $q, r \in \mathbb{Z}$, the right-hand side is an integer, but the left-hand side is not. That is a contradiction, so $4q^2 + 4q + 1$ must not be a multiple of 4.

5. (10 pts) **Extra credit problem.** Prove that

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

Hint: some kind of induction comes to mind.

First, the problem does not say, but the notation implies that n is an integer and $n \geq 1$. So we need to prove that the two sides are equal for every $n \in \mathbb{Z}^+$.

We proved in class that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for every $n \in \mathbb{Z}^+$. Therefore

$$(1 + 2 + 3 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}.$$

If we can prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

then we are done. We will do this by induction. The base case is $n = 1$ and it is indeed true that

$$\frac{1^2(1+1)^2}{4} = 1 = 1^3.$$

Let us suppose that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for some $n \in \mathbb{Z}^+$. We now want to show that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}.$$

By the inductive hypothesis,

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{\big(n^2 + 4(n+1)\big)(n+1)^2}{4} \\
&= \frac{(n^2 + 4n + 4)(n+1)^2}{4} \\
&= \frac{(n+2)^2(n+1)^2}{4}.
\end{aligned}$$

This is what we wanted to show.