MCS 150 Exam 2 Solutions

1. (5 pts each)

   (a) Give an example of sets A, B and C such that $A \in B$, and $B \in C$, and $A \in C$.

   Here is one such example:

   $$A = \{1\}$$
   $$B = \{A\} = \{\{1\}\}$$
   $$C = \{A, B\} = \{\{1\}, \{\{1\}\}\}$$

   It is clear that $A \in B$, $B \in C$, and $A \in C$ because that is how we constructed B and C.

   (b) Prove or disprove the following statement about arbitrary sets A and B. If you think the statement is true, prove it; if you think it is false, provide a counterexample.

   $$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$$

   This is false in general. For example, if $A = \{1\}$ and $B = \{2\}$ then $A \cup B = \{1, 2\}$ and

   $$\mathcal{P}(A) = \{\emptyset, \{1\}\}$$
   $$\mathcal{P}(B) = \{\emptyset, \{2\}\}$$
   $$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\}$$
   $$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

   It is clear that $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.

   Note that any two sets will work as a counterexample as long as neither is a subset of the other. This is because $\mathcal{P}(A \cup B)$ contains $A \cup B$ as an element, while neither $\mathcal{P}(A)$ nor $\mathcal{P}(B)$ can have $A \cup B$ in them unless $A \cup B$ is a subset of A or a subset of B. And $A \cup B$ can only be a subset of A if $B \subseteq A$, and it can only be a subset of B if $A \subseteq B$.

2. (10 pts) Let A, B, and C be any three sets. Prove that

   $$A \times (B - C) = (A \times B) - (A \times C).$$

   Let the ordered pair (x, y) be any element of $A \times (B - C)$. Then $x \in A$ and $y \in B - C$. Hence $y \in B$ but $y \notin C$. Therefore $(x, y) \in A \times B$ but (x, y) cannot be in $A \times C$, which tells us that $(x, y) \in (A \times B) - (A \times C)$. Since (x, y) was an arbitrary element of $A \times (B - C)$, we can now conclude that $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

   Now, let (x, y) be any element of $(A \times B) - (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. Since $(x, y) \in A \times B$, $x \in A$ and $y \in B$. Because $(x, y) \notin A \times C$, either $x \notin A$ or $y \notin C$, or both. But we already know $x \in A$, so y must not be in C. Hence $y \in B$ but $y \notin C$, which tells us that $y \in B - C$. Therefore $(x, y) \in A \times (B - C)$. Since (x, y) was an arbitrary element of $(A \times B) - (A \times C)$, we can now conclude that $(A \times B) - (A \times C) \subseteq A \times (B - C)$.

   Thus $A \times (B - C) = (A \times B) - (A \times C)$.

3. (10 pts) In a game of Pin the Tail on the Donkey, the target on the donkey consists of three concentric circles. The (blindfolded) contestants take turns pinning the tail on the donkey, and score 10 points for hitting the innermost circle, 7 points for the middle circle, and 4 points for the outermost circle. The game ends when the contestants get tired of playing or the ass gets tired of getting pricked in the, you know, rear end. The scores are added up and the person with the highest total score wins. Show that the total score earned by a player can be any integer $n \geq 14$.

First, notice that it is possible to score 14, 15, 16, 17 the following ways:

$$\begin{aligned}14 &= 10 + 4\\ 15 &= 7 + 4 + 4\\ 16 &= 4 + 4 + 4 + 4\\ 17 &= 10 + 7\end{aligned}$$

This establishes the base case of our proof by strong induction. The inductive hypothesis is that k points can be scored by a player for all integers $k = 14, 15, \ldots, n$ for some integer $n \geq 17$. We will show that $n + 1$ points can also be scored. Note that $n - 3 \geq 14$ as $n \geq 17$. Since $14 \leq n - 3 \leq n$, it is possible to score $n - 3$ points by the inductive hypothesis. Therefore it is also possible to score $n + 1$ points by scoring $n - 3$ points and a 4-point prick. This proves that a player can score any integer $n \geq 14$.

The first version of the exam had $n \geq 10$ in the problem statement. That was my mistake. It is not actually possible to score 13 points. Some of you noticed that was a mistake and that the statement was not true and pointed this out. If you did, I gave you full credit. If you did not notice this and tried to give a proof by induction but could not establish the base case, I of course did not hold that against you. In any case, I graded your exam so that my mistake would not hurt your grade.

4. (a) (4 pts) State the Well-Ordering Principle for the set of positive integers.

   The Well-Ordering Principle says that every nonempty subset of the positive integers $\mathbb{Z}^+$ has a least element.

   (b) (6 pts) Use the Well-Ordering Principle to prove that the set of nonnegative integers $\mathbb{Z}^{\geq 0}$ is well-ordered.

   We need to prove that every nonempty subset of $\mathbb{Z}^{\geq 0}$ has a least element. Let S be a nonempty subset of $\mathbb{Z}^{\geq 0}$. If $0 \in S$, then 0 is clearly the smallest element in S. If $0 \notin S$, then $S \subseteq \mathbb{Z}^+$. Since S is nonempty it must have a least element by the Well-Ordering Principle. In either case, S has a least element.

5. (10 pts) **Extra credit problem.** Remember that we defined the Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ by $F_0 = 0$, $F_1 = 1$ and the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 2.$$

To find the n-th Fibonacci number using this relation, you would first have to calculate $F_2, F_3, \ldots, F_{n-1}$. In fact, it is possible to find the value of F_n directly, without doing all that work, and that is what you will show in this problem.

Let $\phi_1 > \phi_2$ be the two roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

The quadratic formula tells us that

$$\phi_1 = \frac{1 + \sqrt{5}}{2} \qquad \text{and} \qquad \phi_2 = \frac{1 - \sqrt{5}}{2}.$$

Interestingly,

$$F_n = \frac{\phi_1^n - \phi_2^n}{\sqrt{5}}$$

for all $n \in \mathbb{Z}^{\geq 0}$. Prove this by using strong induction. (Hints: You may want to start by calculating the first few Fibonacci numbers, say F_0, F_1, F_2, and F_3 using this formula to get a feel for how it works. Calculating by hand will give you more intuition than punching numbers into a calculator. The actual values of ϕ_1 and ϕ_2 are an important part of this argument, so do not forget that you know them.)

First, notice that

$$\frac{\phi_1^0 - \phi_2^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 = F_0$$

$$\frac{\phi_1^1 - \phi_2^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{1}{2} + \frac{1}{\sqrt{5}}2 - \frac{1}{2} + \frac{1}{\sqrt{5}}2}{\sqrt{5}} = 1 = F_1$$

This establishes the base case of the induction. Suppose

$$F_k = \frac{\phi_1^k - \phi_2^k}{\sqrt{5}}$$

for $k = 0, 1, \ldots, n$ where $n \in \mathbb{Z}^{\geq 1}$. We will prove

$$F_{n+1} = \frac{\phi_1^{n+1} - \phi_2^{n+1}}{\sqrt{5}}.$$

We know

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&= \frac{\phi_1^n - \phi_2^n}{\sqrt{5}} + \frac{\phi_1^{n-1} - \phi_2^{n-1}}{\sqrt{5}} \\
&= \frac{\phi_1^n - \phi_2^n + \phi_1^{n-1} - \phi_2^{n-1}}{\sqrt{5}} \\
&= \frac{\phi_1^n + \phi_1^{n-1} - \phi_2^n - \phi_2^{n-1}}{\sqrt{5}} \\
&= \frac{\phi_1^{n-1}(\phi_1 + 1) - \phi_2^{n-1}(\phi_2 + 1)}{\sqrt{5}}
\end{aligned}$$

Since ϕ_1 and ϕ_2 are roots of the quadratic equation

$$x^2 - x - 1 = 0,$$

they also satisfy

$$x^2 = x + 1.$$

That is

$$\phi_1 + 1 = \phi_1^2 \quad \text{and} \quad \phi_2 + 1 = \phi_2^2.$$

Hence

$$F_{n+1} = \frac{\phi_1^{n-1}\phi_1^2 - \phi_2^{n-1}\phi_2^2}{\sqrt{5}} = \frac{\phi_1^{n+1} - \phi_2^{n+1}}{\sqrt{5}}.$$

By induction,

$$F_n = \frac{\phi_1^n - \phi_2^n}{\sqrt{5}}$$

for all $n \in \mathbb{Z}^{\geq 0}$.

This formula is known as Binet's Formula, named after the French mathematician Jacques Philippe Marie Binet, who first published it in the 19th century, although the formula was known to others well before that time.