MCS 220 Exam 1 Solutions

Oct 11, 2019

1. (5 pts each)

(a) Let F be a field and $a \in F$. Prove that

$$(-1)a = -a.$$

Hint: remember that $-a$ means the additive inverse of a.

$$\begin{aligned}
(-1)a &= (-1)a + 0 && \text{because 0 is the additive identity} \\
&= (-1)a + (a + (-a)) && \text{because } 0 = a + (-a) \\
&= ((-1)a + a) + (-a) && \text{by associativity of addition} \\
&= ((-1)a + 1a) + (-a) && \text{because } a = 1a \\
&= (-1 + 1)a + (-a) && \text{by distributivity} \\
&= 0a + (-a) && \text{because -1 is the additive inverse of 1} \\
&= 0 + (-a) && \text{by another homework exercise, } 0a = 0 \\
&= -a && \text{because 0 is the additive identity}
\end{aligned}$$

Hence $(-1)a = -a$.

(b) Let F be an ordered field and $a \in F$. Prove that

$$0 \leq a^2.$$

If $a > 0$ then we can multiply both sides of the inequality $a > 0$ by a and get the inequality $a^2 > 0a$. We know $0a = 0$ by another HW exercise. So $a > 0$ in this case.
If $a = 0$, then $a^2 = 0^2 = 0$.
If $a < 0$, then $0 < -a$ by another homework problem. Now, multiply both sides of the last inequality by $-a$ to get $0(-a) < (-a)(-a)$. We know $0(-a) = 0$. Notice

$$\begin{aligned}
(-a)(-a) &= \big((-1)a\big)\big((-1)a\big) && \text{by part (a)} \\
&= \big(a(-1)\big)\big((-1)a\big) && \text{by commutativity of multiplication} \\
&= \Big(\big(a(-1)\big)(-1)\Big)a && \text{by associativity of multiplication} \\
&= \Big(a\big((-1)(-1)\big)\Big)a && \text{by associativity of multiplication} \\
&= \Big(a\big(-(-1)\big)\Big)a && \text{by part (a), } (-1)(-1) = -(-1) \\
&= (a1)a && \text{since } -(-1) = 1 \text{ by HW exercise } -(-a) = a \\
&= aa && \text{because 1 is the multiplicative identity}
\end{aligned}$$

Hence $(-a)^2 = a^2$. So $0 < (-a)^2$ implies $0 < a^2$.
In all three cases, we found $a^2 \geq 0$.

2. (5 pts each) Let S and T be nonempty sets of real numbers and define

$$S + T = \{s + t \mid s \in S, t \in T\}.$$

Suppose S and T are both bounded from above. In this problem, you will prove that

$$\sup(S + T) = \sup(S) + \sup(T).$$

(a) First, let $\beta = \sup(S)$ and $\gamma = \sup(T)$. Show that $\beta + \gamma$ is an upper bound for $S + T$, that is every $x \in S + T$ satisfies $x \leq \beta + \gamma$.

Let x be any element in $S + T$. Then $x = s + t$ for some $s \in S$ and $t \in T$. Since $\beta = \sup(S)$, we know $s \leq \beta$. Similarly, $t \leq \gamma$. Now, it is easy to see that $s + t \leq \beta + \gamma$. Hence $x \leq \beta + \gamma$ for every $x \in S + T$.

(b) Now prove that if $\epsilon > 0$, then $\beta + \gamma - \epsilon$ cannot be an upper bound of $S + T$ because there must exist some $x \in S + T$ such that $x > \beta + \gamma - \epsilon$. Hint: to find such an x, use the fact that $\beta - \epsilon/2$ cannot be an upper bound of S and $\gamma - \epsilon/2$ cannot be an upper bound of T.

Suppose $\epsilon > 0$. Then $\epsilon/2 > 0$ as well. Therefore $\beta - \epsilon/2$ cannot be an upper bound for S and there must exist some $s_0 \in S$ such that $s_0 > \beta - \epsilon/2$ (by Theorem 1.1.3). Similarly, there must be a $t_0 \in S$ such that $t_0 > \gamma - \epsilon/2$. It is now easy to see that

$$\beta + \gamma - \epsilon = (\beta - \epsilon/2) + (\gamma - \epsilon/2) < s_0 + t_0 \in S + T,$$

and hence $\beta + \gamma - \epsilon$ cannot be an upper bound of $S + T$.

3. (5 pts each) Let F be an ordered field and S a nonempty subset of F.

(a) State the definitions of the supremum and infimum of S.

An element $\beta \in F$ is a supremum of S if

(i) $x \leq \beta$ for all $x \in S$,

(ii) for any $\gamma \in F$ such that $\gamma < \beta$ there exists some $x_0 \in S$ such that $x_0 > \gamma$.

Similarly, an element $\beta \in F$ is an infimum of S if

(i) $x \geq \beta$ for all $x \in S$,

(ii) for any $\gamma \in F$ such that $\gamma > \beta$ there exists some $x_0 \in S$ such that $x_0 < \gamma$.

(b) Suppose S has a supremum β. Define

$$-S = \{-x \mid x \in S\}.$$

Prove that $-\beta$ is the infimum of $-S$.

First, we will show that $-\beta$ is a lower bound for $-S$. Let y be any element of $-S$. Then $y = -x$ for some $x \in S$. We know that $x \leq \beta$. Multiplying this inequality by -1 yields $-\beta \leq -x = y$. This is true for any $y \in -S$, so $-\beta$ is a lower bound of $-S$.
Now, suppose $\gamma > -\beta$. We will show that γ is not a lower bound of $-S$. Multiplying the previous inequality by -1 gives $-\gamma < \beta$. Since β is the supremum of S, there must exist some $x_0 \in S$ such that $-\gamma < x_0$. Multiply this by -1 to get $\gamma > -x_0 \in -S$. So γ is not a lower bound of $-S$.

4. (10 pts) Use induction to prove that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all $n \in \mathbb{Z}^+$.

Note that when $n = 1$, we have $1 = 1$. This establishes the base case. Now, assume that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for some $n \in \mathbb{Z}^+$. Then

$$1 + 3 + 5 + \cdots + (2(n + 1) - 1) = n^2 + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2.$$

Therefore

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all $n \in \mathbb{Z}^+$.

5. (5 pts each) **Extra credit problem.** In this exercise, you will prove that the field of complex numbers cannot be ordered.

   (a) First, prove that in any ordered field F,

   $$-1 < a^2$$

   for all $a \in F$.

   By a homework problem, $0 < 1$. We can add -1 to both sides and we get $-1 < 1 + (-1) = 0$. Now, let $a \in F$. In problem 1, we showed that $0 \leq a^2$. If $0 < a^2$, then $-1 < a^2$ by transitivity. If $a^2 = 0$, then $-1 < 0 = a^2$. In either case, $-1 < a^2$.

   (b) Now, find a complex number z such that no matter how $<$ is defined on $\mathbb{C}$

   $$-1 < z^2$$

   cannot be true. Conclude that it is therefore impossible to define an order $<$ on $\mathbb{C}$ which would satisfy the properties of an ordered field.

   Since $-1 = i^2$, it cannot be true that $-1 < i^2$, no matter how one tries to define $<$ on $\mathbb{C}$. Now, if $<$ were an order on $\mathbb{C}$, it would have to be true that $-1 < z^2$ for all $z \in \mathbb{C}$, including $z = i$. And since that is not true, there must not exist an order on $\mathbb{C}$.