MCS 220 Exam 2 Solutions

Nov 8, 2019

1. (10 pts) Let $a \geq 2$ be an integer. Show by induction (on n) that if n is a nonnegative integer then $n = aq + r$ for some integers q and r such that $0 \leq r < a$.

   Suppose $a \geq 2$ is an integer.
   Base case: if $n = 0$ then $q = r = 0$ work because $0 = a0 + 0$, and $0 \in \mathbb{Z}$ and $0 \leq 0 < a$.
   Now, assume that for some $n \in \mathbb{Z}^{\geq 0}$, there exist $q, r \in \mathbb{Z}$ such that $0 \leq r < a$ and $n = aq + r$. We need to show that $n + 1 = aq' + r'$ for some $q', r' \in \mathbb{Z}$ such that $0 \leq r' < a$. We know

   $$n + 1 = (aq + r) + 1 = aq + r + 1.$$

   Now, if $0 \leq r < a - 1$ then $0 \leq r + 1 < a$, so $n + 1 = aq' + r'$ for $q' = q$ and $r' = r + 1$ is of the right form. If $r = a - 1$ then $r + 1 = a$, and so

   $$n + 1 = aq + r + 1 = aq + a = a(q + 1),$$

   which is of the right form for $q' = q + 1$ and $r' = 0$.
   Therefore the statement is true for any $n \in \mathbb{Z}^{\geq 0}$.

2. (10 pts) Show that the intersection of finitely many open sets in $\mathbb{R}$ is open.

   Let $S_1, \ldots, S_n$ be open sets. If $x \in S_1 \cap \cdots \cap S_n$, then $x \in S_i$ for each $1 \leq i \leq n$. Since S_i is open for each i, there must exist an $\epsilon_i > 0$ such that the ϵ_i-neighborhood of x is in S_i. Let $\epsilon = \min_i(\epsilon_i)$. Then the ϵ-neighborhood of x is contained in the ϵ_i-neighborhood of x for each i, and hence it is contained in S_i. It follows that the ϵ-neighborhood of x is also in $S_1 \cap \cdots \cap S_n$. Therefore x is an interior point of the intersection. Since x was any point in the intersection, every point of the intersection is interior, and hence the intersection is open.

3. (5 pts each)
   (a) Let $S \subseteq \mathbb{R}$. State the definitions of limit point and boundary point of S.

   The point $x \in \mathbb{R}$ is a limit point of S is every ϵ-neighborhood of x contains some $y \neq x$ in S. And x is a boundary point of S if every ϵ-neighborhood of x contains some point in S and another point not in S.

   (b) Give an example of a set $S \subseteq \mathbb{R}$ which has a limit point that is not a boundary point and a boundary point which is not a limit point.

   Let $S = (0, 1) \cup \{2\}$. Then $1/2$ is a limit point, since every ϵ-neighborhood of $1/2$ will contain some $x \in S$ other than $1/2$, e.g. $3/4$ if $\epsilon > 1/4$ or $x = 1/2 + \epsilon/2$ if $\epsilon \leq 1/4$. But $1/2$ is not a boundary point because for example, the $1/2$-neighborhood of $1/2$ contains no point that is not is S. On the other hand, 2 is a boundary point, since every ϵ-neighborhood of 2 contains some $x \notin S$, e.g. $x = 2 + \epsilon/2$, while it also contains $2 \in S$. But 2 is not a limit point because for example, the $1/2$-neighborhood of 2 contains no point of S other than 2 itself.

4. (10 pts) Prove that a set S of real numbers is closed if and only if contains all of its limit points.

   First, suppose that $S \subseteq \mathbb{R}$ is closed. Let $x \in S^c$. Since S is closed, S^c is open, so x must be an interior point of S^c. This means that some ϵ-neighborhood of x lies in S. But such an

ϵ-neighborhood of x contains no point of S, so x cannot be a limit point of S. This shows that no limit point of S can be in S^c, that is every limit point of S must be in S.

Conversely, suppose that S contains all of its limit points. We will show that every point of S^c is interior. Let $x \in S^c$. Since $x \notin S$, it cannot be a limit point of S, so there must be some ϵ-neighborhood of x that contains no point of S other than x itself. But x itself is also in S^c, so every point of such an ϵ-neighborhood is in S^c, which makes x an interior point in S^c. We have just proved that any point $x \in S^c$ must be interior, hence S^c is open and S is closed.

5. (10 pts) **Extra credit problem.** For a set S of real numbers, we defined the closure $\overline{S}$ of S as the union of S and its boundary:
$$\overline{S} = S \cup \partial S.$$
Let S' be the set of all limit points of S. Show that
$$\overline{S} = S \cup S',$$
that is we could have also defined the closure of S as the union of S and its limit points. (Hint: remember that two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.

We need to show that $S \cup \partial S \subseteq S \cup S'$ and $S \cup S' \subseteq S \cup \partial S$. Suppose $x \in S \cup \partial S$. If $x \in S$ then x is also in $S \cup S'$. If $x \notin S$ then x must be in ∂S, so x is a boundary point of S. Then every ϵ-neighborhood of x contains a point $y \in S$. Since $x \notin S$, such a y must be different from x. So every ϵ-neighborhood of x contains a point in S other than x itself. Hence x is a limit point of S. That is $x \in S'$, and hence $x \in S \cup S'$. So every $x \in S \cup \partial S$ is also in $S \cup S'$. This shows $S \cup \partial S \subseteq S \cup S'$.

Now, suppose $x \in S \cup S'$. If $x \in S$ then x is also in $S \cup \partial S$. If $x \notin S$ then x must be in S', so x is a limit point of S. Then every ϵ-neighborhood of x contains a point $y \neq x$ in S. Hence every ϵ-neighborhood of x contains a point in S. But every ϵ-neighborhood of x also contains x, which is a point not in S. Hence x is a boundary point of S. That is $x \in \partial S$, and hence $x \in S \cup \partial S$. So every $x \in S \cup S'$ is also in $S \cup \partial S$. This shows $S \cup S' \subseteq S \cup \partial S$. We can now conclude that $S \cup \partial S = S \cup S'$