MCS 220 Final Exam Solutions

Dec 18, 2019

1. Let S and T be nonempty sets of real numbers and define

$$S - T = \{s - t \mid s \in S, t \in T\}.$$

Show that if S is bounded from above and T is bounded from below, then

$$\sup(S - T) = \sup(S) - \inf(T).$$

(Hint: For $\epsilon > 0$, prove that $\sup(S) - \inf(T) - \epsilon$ cannot be an upper bound of $S - T$ because $\sup(S) - \epsilon/2$ is not an upper bound of S and $\inf(T) + \epsilon/2$ is not a lower bound of T.)

Since S is bounded from above, it has a supremum β. Similarly, T has an infimum γ. First, we will show that $\beta - \gamma$ is an upper bound for $S - T$. Let $x \in S - T$. Then $x = s - t$ for some $s \in S$ and $t \in T$. We know $s \leq \beta$ and $t \geq \gamma$. So

$$t \geq \gamma \implies -t \leq -\gamma \implies s - t \leq s - \gamma \leq \beta - \gamma$$

using the usual properties of $<$. So $x \leq \beta - \gamma$ for every $x \in S - T$.

Now, to show that $\beta - \gamma$ is also the least upper bound for $S - T$, let $\epsilon > 0$. We will show that $(\beta - \gamma) - \epsilon$ cannot be an upper bound for $S - T$. First, since β is the least upper bound for S, there must be some $s_0 \in S$ such that $s_0 > \beta - \epsilon/2$. Similarly, there must be a $t_0 \in T$ such that $t_0 < \gamma + \epsilon/2$. Hence

$$t_0 < \gamma + \frac{\epsilon}{2} \implies -t_0 > -\gamma - \frac{\epsilon}{2} \implies s_0 - t_0 > s_0 - \gamma - \frac{\epsilon}{2} \leq \beta - \frac{\epsilon}{2} - \gamma - \frac{\epsilon}{2} = (\beta - \gamma) - \epsilon.$$

So $x_0 = s_0 - t_0$ is an element of $S - T$ that is larger than $(\beta - \gamma) - \epsilon$. Such an element of $S - T$ exists for each $\epsilon > 0$. Hence $\beta - \gamma$ is the supremum of $S - T$ by Theorem 1.1.3.

2. (10 pts) The Fibonacci numbers $\{F_n\}_{n=1}^{\infty}$ are are defined by $F_1 = F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 2.$$

Prove by induction that for $n \geq 1$,

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Note that

$$F_1 = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

and

$$\begin{aligned} F_2 &= \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} \\ &= \frac{1 + 2\sqrt{5} + 5 - (1 - 2\sqrt{5} + 5)}{4\sqrt{5}} \\ &= \frac{2\sqrt{5} + 2\sqrt{5}}{4\sqrt{5}} \\ &= 1 \end{aligned}$$

This establishes the base case for $n = 1$ and $n = 2$. Now, suppose that the statement is true for some $n \geq 2$ and also for $n - 1$. Then

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} + \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} + \frac{2(1+\sqrt{5})^{n-1} - 2(1-\sqrt{5})^{n-1}}{2^n\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^n + 2(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^n - 2(1-\sqrt{5})^{n-1}}{2^n\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-1}(1+\sqrt{5}+2) - (1-\sqrt{5})^{n-1}(1-\sqrt{5}+2)}{2^n\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-1}(3+\sqrt{5}) - (1-\sqrt{5})^{n-1}(3-\sqrt{5})}{2^n\sqrt{5}}.
\end{aligned}$$

Now, notice that

$$\begin{aligned}
(1+\sqrt{5})^2 = 1 + 2\sqrt{5} + 5 = 2(3+\sqrt{5}) &\implies 3 + \sqrt{5} = \frac{(1+\sqrt{5})^2}{2} \\
(1-\sqrt{5})^2 = 1 - 2\sqrt{5} + 5 = 2(3-\sqrt{5}) &\implies 3 - \sqrt{5} = \frac{(1-\sqrt{5})^2}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
F_{n+1} &= \frac{(1+\sqrt{5})^{n-1}\frac{(1+\sqrt{5})^2}{2} - (1-\sqrt{5})^{n-1}\frac{(1-\sqrt{5})^2}{2}}{2^n\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}},
\end{aligned}$$

exactly as we wanted to show.

3. (10 pts) Find

$$\lim_{x \to 4} \sqrt{x}$$

and justify your answer with an $\epsilon - \delta$ proof.

We will show that

$$\lim_{x \to 4} \sqrt{x} = 2.$$

Let $\epsilon > 0$. We need to prove that there is a $\delta > 0$ such that $|\sqrt{x} - 2| < \epsilon$ whenever $0 < |x - 4| < \delta$. First, notice that

$$x - 4 = \sqrt{x}^2 - 2^2 = (\sqrt{x} + 2)(\sqrt{x} - 2)$$

and so

$$|x - 4| = |(\sqrt{x} + 2)(\sqrt{x} - 2)| = |\sqrt{x} + 2||\sqrt{x} - 2|.$$

Now, if x is close to 4, then $\sqrt{x} \geq 0$, so $\sqrt{x} + 2 \geq 2$. Hence we can divide by $\sqrt{x} + 2$, as it is not 0, to get

$$|\sqrt{x} - 2| = \frac{|x - 4|}{|\sqrt{x} + 2|}.$$

Now if $|x - 4| < \delta$ then

$$|\sqrt{x} - 2| = \frac{|x - 4|}{|\sqrt{x} + 2|} < \frac{\delta}{|\sqrt{x} + 2|}.$$

Suppose $\delta \leq 1$, that is $|x-4|< 1$. Then $3 < x < 5$ and since the function $g(x) = \sqrt{x}$ is an increasing function, $\sqrt{3} < \sqrt{x} < \sqrt{5}$. Hence $\sqrt{3}+2 < \sqrt{x}+2 < \sqrt{5}+2$. In particular, $\sqrt{x}+2$ is positive and therefore $|\sqrt{x}+2|= \sqrt{x}+2$. So

$$\frac{\delta}{|\sqrt{x}+2|} = \frac{\delta}{\sqrt{x}+2} \leq \frac{\delta}{|\sqrt{3}+2|}.$$

This suggests that setting $\delta = \min(1, (\sqrt{3}+2)\epsilon)$ should be a good value for δ.

So let $\delta = \min(1, (\sqrt{3}+2)\epsilon)$. Then

$$\begin{aligned} &\delta \leq 1 \\ &\delta \leq (\sqrt{3}+2)\epsilon \end{aligned}$$

are both true. Hence if $|x-4|< \delta$, then

$$\begin{aligned} |\sqrt{x}-2| &= \frac{|x-4|}{|\sqrt{x}+2|} \\ &< \frac{\delta}{|\sqrt{x}+2|} \\ &\leq \frac{\delta}{|\sqrt{3}+2|} \\ &\leq \frac{(\sqrt{3}+2)\epsilon}{|\sqrt{3}+2|} \\ &= \epsilon. \end{aligned}$$

4. Let S be a subset of the real numbers.
   (a) (2 pts) Define what a limit point of S is.

   The number $x \in \mathbb{R}$ is a limit point of S if every deleted neighborhood of x contains some point in S.

   (b) (8 pts) Prove that S is closed if and only it contains all of its limit points.

   First, let S be closed. Then S^c is open. If $x \in S^c$ then x must be an interior point of S^c, that is there is a neighborhood of x that lies in S^c. Such a neighborhood of x contains no point in S, and therefore neither does the corresponding deleted neighborhood with x removed. Therefore x cannot be a limit point of S. We have just shown that no point of S^c can be a limit point of S. Therefore any limit point of S must be in S.
   Conversely, suppose that S contains all of its limit points. We will show that S^c is open and hence S is closed. Let $x \in S^c$. Since x is not in S, x cannot be a limit point of S. Therefore there must exist a deleted neighborhood U of x which contains no point in S. Since x itself is not in S, we can add x to U, and the neighborhood $U \cup \{x\}$ of x will not contain any point of S. Therefore $U \cup \{x\}$ lies in S^c. Thus x is an interior point of S^c. We have just shown that any point $x \in S^c$ is an interior point. Therefore S^c is an open set.

5. (a) (4 pts) Let f be a function of real numbers and $x_0 \in \mathbb{R}$. Define what it means for f to be left and right continuous at x_0.

   The function f is left continuous at x_0 if

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

and right continuous at x_0 if

$$\lim_{x \to x_0^+} f(x) = f(x_0).$$

(b) (6 pts) Give an example of a function f and $x_0 \in \mathbb{R}$ such that f is left continuous but not right continuous at x_0. Be sure to fully justify your example.

One such example would be the piecewise-defined function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Then

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0 = f(0),$$

so f is left continuous at 0. On the other hand,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1 \neq f(0),$$

so f is not right continuous at 0.

6. (10 pts) Let f be a function of real numbers and x_0 an interior point of its domain. Prove that if f is differentiable at x_0 then there exists a function E defined on some neighborhood of x_0 such that

$$E(x_0) = 0 \qquad \text{and} \qquad \lim_{x \to x_0} E(x) = 0,$$

and

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + E(x)(x - x_0).$$

(Hint: Use the equation given for $f(x)$ to define $E(x)$.)

Suppose f is a function differentiable at x_0. Then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. This means that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon$$

for every x such that $0 < |x - x_0| < \delta$. In particular, this means that the number

$$\frac{f(x) - f(x_0)}{x - x_0}$$

exists whenever $0 < |x - x_0| < \delta$. Pick any one such δ. For x in the δ-neighborhood of x_0, define

$$E(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases}.$$

First, observe that if $x \neq x_0$ then

$$\begin{aligned} & f(x_0) + f'(x_0)(x - x_0) + E(x)(x - x_0) \\ &\quad = f(x_0) + f'(x_0)(x - x_0) + \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right)(x - x_0) \\ &\quad = f(x_0) + f'(x_0)(x - x_0) + f(x) - f(x_0) - f'(x_0)(x - x_0) \\ &\quad = f(x) \end{aligned}$$

And if $x = x_0$, then

$$f(x_0) + f'(x_0)(x - x_0) + E(x)(x - x_0) = f(x_0) + f'(x_0)(x_0 - x_0) + 0(x_0 - x_0) = f(x_0).$$

So

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + E(x)(x - x_0)$$

does indeed hold for any x wherever $E(x)$ is defined.

By definition, $E(x_0) = 0$. The only thing that remains to show is that

$$\lim_{x \to x_0} E(x) = 0.$$

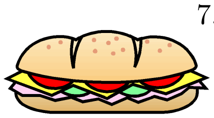
Since

$$E(x) = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)$$

whenever $x \neq x_0$,

$$\begin{aligned}
\lim_{x \to x_0} E(x) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - \lim_{x \to x_0} f'(x_0) \\
&= f'(x_0) - f'(x_0) \\
&= 0
\end{aligned}$$

by using the limit laws (Theorem 2.1.4) and the fact that the limit of a constant function is just the value of that function.



7. **Extra credit problem.** The Squeeze or Sandwich Theorem, which you may well have learned in your calculus class, says that if f, g, and h are functions and x_0 is an interior point of their domains such that $f(x) \leq g(x) \leq h(x)$ for every x in some deleted neighborhood of x_0 and

$$L = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x)$$

then

$$\lim_{x \to x_0} g(x) = L.$$

(a) (10 pts) Use a $\delta - \epsilon$ argument to prove the Squeeze Theorem. (Hint: for an $\epsilon > 0$, start by showing there is a $\delta > 0$ such that if $x \neq x_0$ is closer to x_0 than a distance of δ then $f(x)$ and $h(x)$ are both within a distance of ϵ of L while $g(x)$ must be between them.)

Let $\epsilon > 0$. We need to show that there is a $\delta > 0$ such that

$$|g(x) - L| < \epsilon$$

whenever

$$0 < |x - x_0| < \delta.$$

First, since

$$L = \lim_{x \to x_0} f(x)$$

there is a $\delta_1 > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever

$$0 < |x - x_0| < \delta_1.$$

Similarly, there is a $\delta_2 > 0$ such that

$$|h(x) - L| < \epsilon$$

whenever

$$0 < |x - x_0| < \delta_2.$$

Finally, it was given that there is a $\delta_3 > 0$ such that

$$f(x) \le g(x) \le h(x)$$

whenever

$$0 < |x - x_0| < \delta_3.$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$ and let $0 < |x - x_0| < \delta$. Then

$$0 < |x - x_0| < \delta_1$$
$$0 < |x - x_0| < \delta_2$$
$$0 < |x - x_0| < \delta_3$$

are all true. Hence

$$|f(x) - L| < \epsilon \implies L - \epsilon < f(x) < L + \epsilon$$

and

$$|h(x) - L| < \epsilon \implies L - \epsilon < h(x) < L + \epsilon$$

and

$$f(x) \le g(x) \le h(x)$$

must also be true. Therefore

$$\implies L - \epsilon < f(x) \le g(x) \le h(x) < L + \epsilon,$$

which shows that $L - \epsilon < g(x) < L + \epsilon$. That is $|g(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta$.

(b) (5 pts) Instead of $L = \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x)$, suppose that

$$L_1 = \lim_{x \to x_0} f(x) \qquad \text{and} \qquad L_2 = \lim_{x \to x_0} h(x)$$

both exist and $L_1 \le L_2$. Can you conclude that

$$L_1 \le \lim_{x \to x_0} g(x) \le L_2?$$

If so, give a proof; if not, find a counterexample.

This is not true. For example, let

$$f(x) = -1$$
$$h(x) = 1$$
$$g(x) = \sin\left(\frac{1}{x}\right)$$

It is clear for any $x \ne 0$ that $f(x) \le g(x) \le h(x)$. Also

$$\lim_{x \to 0} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to 0} h(x) = 1$$

but we saw in class that

$$\lim_{x \to 0} g(x)$$

does not exist because no matter how close we look to 0, there will always be values of x_1 and x_2 for which $g(x_1) = -1$ and $g(x_2) = 1$.