MCS 221 Exam 1 Solutions

1. (10 pts) Let F be any field. Show that $(ab)x = a(bx)$ for all $x \in F^n$ and all $a, b \in F$.

   Let $x \in F^n$. So $x = (x_1, x_2, \ldots, x_n)$ where $x_i \in F$ for all i. Now

$$\begin{aligned}(ab)x &= (ab)(x_1, x_2, \ldots, x_n)\\ &= \big((ab)x_1, (ab)x_2, \ldots, (ab)x_n\big)\end{aligned}$$

   Also

$$\begin{aligned}a(bx) &= a(b(x_1, x_2, \ldots, x_n))\\ &= a(bx_1, bx_2, \ldots, bx_n)\\ &= \big(a(bx_1), a(bx_2), \ldots, a(bx_n)\big)\end{aligned}$$

   Since multiplication in F is associative, $(ab)x_i = a(bx_i)$ for all i. Hence

$$(ab)x = \big((ab)x_1, (ab)x_2, \ldots, (ab)x_n\big) = \big(a(bx_1), a(bx_2), \ldots, a(bx_n)\big) = a(bx).$$

2. (10 pts) Let V be a vector space. Suppose that U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

   Since U_1 and U_2 are subspaces, $0 \in U_1$ and $0 \in U_2$. Hence $0 \in U_1 \cap U_2$. Now, let $u, v \in U_1 \cap U_2$. Then $u, v \in U_1$, and $u + v \in U_1$ because U_1 is closed under addition. Similarly, $u, v \in U_2$, and therefore $u + v \in U_2$. So $u + v \in U_1 \cap U_2$. Finally, let $u \in U_1 \cap U_2$ and $a \in F$. Then $u \in U_1$, and $au \in U_1$ because U_1 is closed under scalar multiplication. Similarly, $u \in U_2$, and therefore $au \in U_2$. So $au \in U_1 \cap U_2$. Therefore $U_1 \cap U_2$ is a subspace of V.

3. (5 pts each)
   (a) We saw in class that $V = \mathbb{C}$ is a vector space over the field $F = \mathbb{R}$. Let

$$U = \{x + xi \mid x \in \mathbb{R}\}.$$

   Show that U is a subspace of V.

   Note $0 = 0 + 0i \in U$. Now, let $u, v \in U$. Then $u = x + xi$ for some $x \in \mathbb{R}$ and $v = y + yi$ for some $y \in \mathbb{R}$.

$$u + v = (x + xi) + (y + yi) = (x + y) + (x + y)i,$$

   which is also in U since its real and imaginary parts are the same. Finally, let $u \in U$ and $a \in \mathbb{R}$. Then $u = x + xi$ for some $x \in \mathbb{R}$ and $au = a(x + xi) = ax + axi$, which is also in U since its real and imaginary parts are the same. Hence U is a subspace of V.

   (b) We also know that $V = \mathbb{C}$ is a vector space over the field $F = \mathbb{C}$. Once again, let

$$U = \{x + xi \mid x \in \mathbb{R}\}.$$

   Is U a subspace of V?

   No it is not. It is not closed under scalar multiplication. For example, choose $u = 1 + i \in U$ and $a = i \in \mathbb{C}$. Then

$$au = i(1 + i) = i + i^2 = -1 + i,$$

   which is not in U because the real part is -1 and the imaginary part is 1.

4. (10 pts)Let V be a vector space over some field F, and let $U_1, U_2, \ldots, U_m$ be subspaces of V. Prove that the sum $U_1 + U_2 + \cdots U_m$ is a subspace of V, and that it is the smallest subspace of V that contains $U_1, U_2, \ldots, U_m$.

   See Theorem 1.39 in your textbook or the proof in your lecture notes.

5. (10 pts) **Extra credit problem.** Let $V = \mathbb{R}^{\mathbb{R}}$, the set of all functions $f : \mathbb{R} \to \mathbb{R}$. We saw in class that V is a vector space over $\mathbb{R}$. Recall from calculus class that a function $f : \mathbb{R} \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Let $U \subset V$ be the set of all even functions and $W \subset V$ be the set of all odd functions. We noted in class that U and W are subspaces of V. Prove that $U + W = V$. (Hint: If $f : \mathbb{R} \to \mathbb{R}$ is any function, convince yourself that $g(x) = f(x) + f(-x)$ is always an even function. Can you find a similar way to construct an odd function from f? Does this help you write any function f as a sum of an even function and an odd function?)

   Let f be any function in V. Let $g(x) = f(x) + (f - x)$. Notice that

$$g(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = g(x)$$

for any $x \in \mathbb{R}$, so g is an even function. Let $h(x) = f(x) - f(-x)$. Then

$$h(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -\big(f(x) - f(-x)\big) = -h(x)$$

for any $x \in \mathbb{R}$, so h is an odd function. Now let $u(x) = \frac{f(x)+f(-x)}{2}$ and $w(x) = \frac{f(x)-f(-x)}{2}$. Then

$$u(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = u(x)$$

$$w(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = w(x)$$

for any $x \in \mathbb{R}$. So $u \in U$ and $w \in W$. Notice $u + w = f$ since

$$u(x) + w(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{2f(x)}{2} = f(x),$$

and hence $f \in U + W$. We have just shown that any $f \in V$ is in $U + W$, so $V \subseteq U + W$. Since U and W are subspaces of V, we know that $U + W \subseteq V$ (e.g. by problem 4 on this exam), so $V = U + W$.