MCS 221 Exam 2 Solutions

Nov 16, 2018

1. (10 pts) Suppose v_1, v_2, v_3, v_4 spans the vector space V. Prove that the list
$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$
also spans V.

   Let $v \in V$ be any vector. Since v_1, v_2, v_3, v_4 spans V, there exist scalars $a_1, \dots, a_4$ such that
$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4.$$
Substitute
$$\begin{aligned} v_3 &= (v_3 - v_4) + v_4 \\ v_2 &= (v_2 - v_3) + (v_3 - v_4) + v_4 \\ v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4 \end{aligned}$$
into the expression above to get
$$\begin{aligned} v &= a_1\big((v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4\big) \\ &\quad + a_2\big((v_2 - v_3) + (v_3 - v_4) + v_4\big) + a_3\big((v_3 - v_4) + v_4\big) + a_4v_4 \\ &= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) \\ &\quad + (a_1 + a_2 + a_3 + a_4)v_4, \end{aligned}$$
which is a linear combination of $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$. Hence any $v \in V$ is in the span of $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$.

2. (10 pts) Recall that $\mathcal{P}_m(F)$ is the vector space of polynomials of degree m or less over the field F. Suppose $p_0, p_1, \dots, p_m$ are polynomials in $\mathcal{P}_m(F)$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \dots, p_m$ is not linearly independent.

   Notice that if q is a linear combination of $p_0, p_1, \dots, p_m$, that is $q = a_0p_0 + a_mp_1 \cdots + a_mp_m$ then
$$q(2) = a_0p_0(2) + a_1p_1(2) \cdots + a_mp_m(2) = 0.$$
Now, let $q(x) = 1$. Since $q(2) \neq 0$, q is not a linear combination of $p_0, p_1, \dots, p_m$. Consider the extended list $p_0, p_1, \dots, p_m, q$. This list is too long to be linearly independent because it has $m+2$ elements, while $1, x, x^2, \dots, x^m$ is a list of $m+1$ elements and it spans $\mathcal{P}_m(F)$, as we saw in class, and we know that a linearly independent list cannot be longer than a spanning list. So $p_0, p_1, \dots, p_m, q$ is linearly dependent. by the Linear Dependence Lemma, one of the polynomials must be a linear combination of the preceding ones. We already noted that q cannot be that polynomial, so it must be p_j for some j. But then $p_0, p_1, \dots, p_m$ is linearly dependent as the list contains a polynomial that is a linear combination of the others.

3. (10 pts) Solve the following system of linear equations by row-reducing its augmented matrix.
$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -1 \\ -7x_1 + 14x_2 + 6x_3 &= 37 \\ 3x_1 - 6x_2 - 4x_3 &= -13 \end{aligned}$$

$$
\begin{pmatrix} 1 & -2 & -3 & -1 \\ -7 & 14 & 6 & 37 \\ 3 & -6 & -4 & -13 \end{pmatrix}
\xrightarrow{7R_1+R_2\to R_2}
\begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 0 & -15 & 30 \\ 3 & -6 & -4 & -13 \end{pmatrix}
$$

$$
\xrightarrow{R_3-3R_1\to R_3}
\begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 0 & -15 & 30 \\ 0 & 0 & 5 & -10 \end{pmatrix}
$$

$$
\xrightarrow{-\frac{1}{15}R_2\to R_2}
\begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 5 & -10 \end{pmatrix}
$$

$$
\xrightarrow{R_3-5R_2\to R_3}
\begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\xrightarrow{3R_2+R_1\to R_1}
\begin{pmatrix} 1 & -2 & 0 & -7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

So

$$
\begin{aligned}
x_1 - 2x_2 = -7 &\implies x_1 = 2x_2 - 7 \\
x_3 = 2
\end{aligned}
$$

where $x_2 \in F$ is a free variable, that is x_2 is any element of F.

4. (10 pts) Let A be the coefficient matrix of the system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n.
\end{aligned}
$$

Show that if the reduced row-echelon form of A is the identity matrix I_n then A has a left inverse, that is a matrix C such that $CA = I_n$.

This is because every elementary row operation corresponds to multiplication of the augmented matrix by an elementary matrix on the left. For example $7R_1 + R_2 \to R_2$ would correspond to multiplication by

$$
\begin{pmatrix} 1 & & & \\ 7 & 1 & & \\ & & \vdots & \\ & & & 1 \end{pmatrix}.
$$

Now, if there is a sequence of elementary row operations that reduces the coefficient matrix A to I_n, and the corresponding elementary matrices are $E_1, E_2, \ldots, E_k$, then

$$
E_k E_{k-1} \cdots E_1 A = I_n.
$$

Hence the matrix $C = E_k E_{k-1} \cdots E_1$ satisfies $CA = I_n$. That is C is a left inverse of A.

5. (10 pts) **Extra credit problem.** Let $V = \mathcal{P}_m(\mathbb{R})$ over the field $\mathbb{R}$ and let

$$
U = \{p \in \mathcal{P}_m(\mathbb{R}) \mid p(2) = 0\}.
$$

It is easy to see that U is a subspace of V. Now, suppose the list $p_1, p_2, \ldots, p_k$ spans U. Show that if $q \in V$ is any polynomial that is not in U, then the list $p_1, p_2, \ldots, p_k, q$ spans V. (Hint: if r is any polynomial in V, first find a scalar c such that the polynomial $r - cq$ has a value of 0 at 2, then express $r - cq$ as a linear combination of $p_1, p_2, \ldots, p_k$.)

Let $p_1, p_2, \ldots, p_k$ and q be as in the problem statement. We need to show that any polynomial $r \in \mathcal{P}_m(\mathbb{R})$ is a linear combination of $p_1, p_2, \ldots, p_k, q$. Since $q \notin U$, we must have $q(2) \neq 0$. Let $a = q(2)$ and $b = r(2)$. Since $a \neq 0$, we can divide b by a. Let $c = b/a$. Now,

$$(r - cq)(2) = r(2) - cq(2) = b - \frac{b}{a}\, a = 0.$$

Hence $r - cq \in U$ and

$$r - cq = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k$$

for some scalars $a_1, \ldots, a_k$. It follows that

$$r = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k + cq \in \operatorname{span}(p_1, p_2, \ldots, p_k, q).$$

This is true for any $r \in V$, therefore $p_1, p_2, \ldots, p_k, q$ spans all of V.